Unit 11

Force

This unit online, with its interactives:
science-playground.com/s/9a11

A bus has a flat tyre on the Araniko Highway. The driver fits a lug wrench onto a nut and pulls, but the nut won't move. So he slips a length of iron pipe over the end of the wrench, pulls again, and the nut turns at once. He didn't pull any harder; he only pulled farther from the nut. Down in Pokhara, a boatman on Phewa Lake steps from his boat onto the shore, and the boat slides backward away from the bank, though nobody pushed it.

Both happen because of rules about force that are older than any bus. A force can make a thing turn, and how well it turns depends on where the force acts as much as on how big it is. A force changes a body's motion, and when two bodies push on each other, the motion one gains the other loses. This unit is about those rules: how to add forces and other quantities that have a direction, how to read a journey from a velocity–time graph, how forces turn things, and how momentum is passed from one body to another.



Figure 11.1 *A boat on Phewa Lake, Pokhara. Push off the shore with a paddle, or step out onto the bank, and the boat moves the other way: you will see why in 11.4.*
Photo: सुयोग सत्याल, CC BY-SA 4.0

11.1 Vector and scalar quantities

Your friend phones from Bhaktapur Durbar Square and says, "Walk 3 km and you will find me." Can you find her? Not yet. Three kilometers north takes you toward

Nagarkot's road, three kilometers west toward Thimi. To find her you need to know **which way** as well as **how far**.

Anything we can measure in physics is a **physical quantity**: length, mass, time, speed, force and so on. Every physical quantity has a **magnitude**, its size, written as a number with a unit (3 km, 50 kg, 20 s). Some quantities need nothing more. Others make sense only with a direction as well.

A scalar quantity is a physical quantity that has magnitude only. Mass, time, temperature, volume and speed are scalars: "50 kg", "20 s" or "30 °C" says everything there is to say.

A vector quantity is a physical quantity that has both magnitude and direction, and that adds by the rules of vector addition below. Displacement, velocity, acceleration and force are vectors: "3 km to the north", "20 m/s toward Pokhara", "a force of 50 N downward".

Scalar quantities	Vector quantities
distance (5 km)	displacement (5 km to the east)
speed (60 km/h)	velocity (60 km/h toward the south)
mass (60 kg)	weight (600 N, downward)
time, temperature, volume, density	acceleration (9.8 m/s², downward)
work, energy, power	force (20 N to the left)
pressure, electric current, charge	momentum (section 11.4)

Distance and displacement show the difference well. A bus goes from Kathmandu to Pokhara and back, about 200 km each way. The distance it covers is 400 km, a scalar. Its displacement, the straight-line change in its position with a direction, is **zero**: it ended where it started.

CAREFUL

Three that fool people

Electric current flows in a direction along a wire, but it is a scalar: two currents meeting at a junction simply add as numbers, whatever the angle between the wires, so they don't follow vector addition. **Pressure** pushes equally in every direction at a point in a liquid or gas, so it has no single direction: a scalar. **Weight** is a vector (a force, always pointing down toward the Earth's center), while **mass** is a scalar.

Representing a vector

A vector is drawn as an **arrow**. The arrow points in the vector's direction, and its length shows its magnitude, drawn to a chosen **scale**. With a scale of 1 cm = 10 N, a force of 40 N to the east is an arrow 4 cm long pointing east. The end the arrow starts from is its **tail**, the pointed end its **head**.

In print, a vector is written in bold, **F**, or with a small arrow over its letter, $\vec{F}$; its magnitude alone is written in ordinary type, F, or as $|\vec{F}|$.

Two vectors are **equal** when they have the same magnitude and the same direction, wherever they are drawn. The **negative** of a vector, $-\mathbf{A}$, has the same magnitude as **A** but points the opposite way.

Check 1

Which of these is a vector quantity?

a the mass of a sack of rice

b the time a bus takes to reach Dhulikhel

c the weight of a sack of rice

d the speed of a bus

Addition of vectors

Scalars add like ordinary numbers: 2 kg of rice and 3 kg of rice make 5 kg, always. Vectors don't. Walk 4 km east and then 3 km north, and you have walked 7 km, but you are only 5 km from where you started. The single vector that has the same effect as two or more vectors together is called their **resultant**.

Vectors along one line

When two vectors act **in the same direction**, the resultant is their sum, in that direction. Two porters pushing a cart forward with 40 N and 30 N give it a resultant push of 70 N forward.

When two vectors act **in opposite directions**, the resultant is their difference, in the direction of the larger one. In a tug of war, if one team pulls with 1,500 N to the left and the other with 1,350 N to the right, the resultant is 150 N to the left, and the rope moves left.

The triangle law

When the vectors are at an angle, draw them **head to tail**.

Triangle law of vector addition: if two vectors are represented in magnitude and direction by the two sides of a triangle taken in order (the tail of the second at the head of the first), their resultant is represented in magnitude and direction by the third side, drawn from the tail of the first to the head of the second.

Walk 4 km east from A to B, then 3 km north from B to C. The two walks are the sides AB and BC, head to tail; your resultant displacement is AC, from where you started to where you ended.

The parallelogram law

The same rule can be drawn with both vectors starting from one point, as two forces pulling on the same object do.

Parallelogram law of vector addition: if two vectors acting at a point are represented in magnitude and direction by the two adjacent sides of a parallelogram drawn from that point, their resultant is represented in magnitude and direction by the diagonal of the parallelogram from the same point.

The two laws always give the same answer: the far side of the parallelogram is the second vector moved to the head of the first, so the triangle is just half of the parallelogram.

Two vectors at right angles

When the two vectors are **perpendicular**, the triangle is a right-angled triangle, and the resultant can be calculated by Pythagoras' theorem instead of being measured on a drawing:

$$R = \sqrt{A^2 + B^2} \qquad \tan\theta = \frac{B}{A}$$

where θ is the angle the resultant makes with **A**.

Two vectors at any angle

At any other angle, draw the vectors to scale, head to tail, and measure the resultant with a ruler and its direction with a protractor. This is the **graphical method**, and it works for any number of vectors: draw them all head to tail, and the resultant runs from the first tail to the last head.

The widget shows a rule worth remembering: the resultant of two vectors is **largest** when they point the same way (R = A + B) and **smallest** when they point opposite ways (R = A − B, if A is the larger). At every other angle it lies between the two.

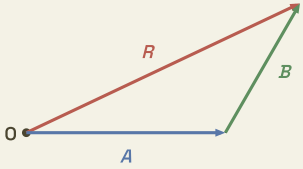
Subtraction of vectors

To subtract a vector, add its negative:

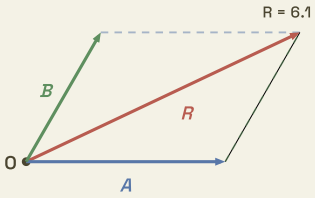
$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

Turn **B** round so it points the opposite way, then add it to **A** by the triangle law, as in the third drawing of Figure 11.2.

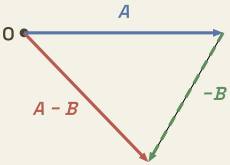
Triangle law



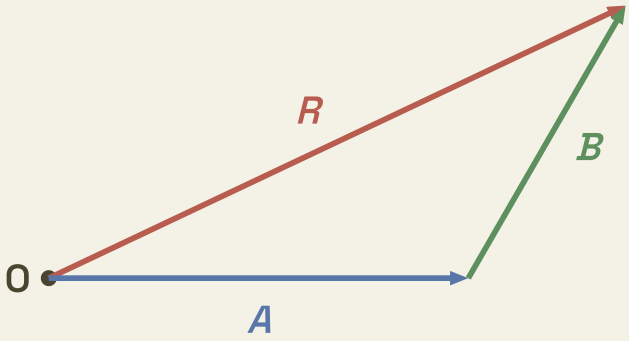
Parallelogram law



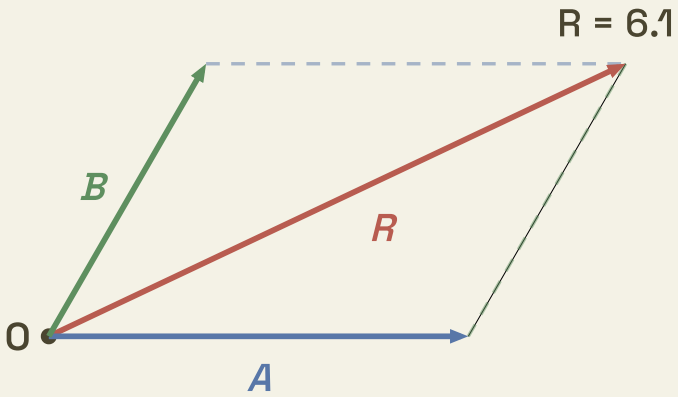
Subtraction



Triangle law



Parallelogram law



Subtraction



Vector subtraction is how we find a **change** in a vector, such as a change in velocity: change in velocity = final velocity − initial velocity. On a line, this is just a matter of signs. A ball hits a wall at 8 m/s and bounces straight back at 6 m/s. Taking "away from the wall" as positive, its velocity changes from −8 m/s to +6 m/s, so the change is $6 - (-8) = \mathbf{14\ m/s\ away\ from\ the\ wall}$, not 2 m/s. That change is what the wall's push produced, and you will need it in 11.4.

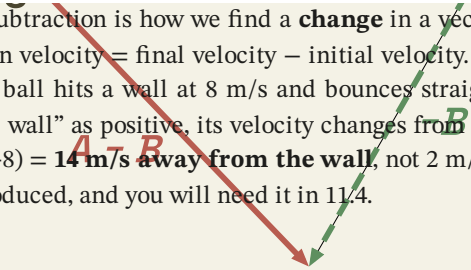


Figure 11.2 *Three ways to draw two vectors **A** (4 units) and **B** (3 units, at 60° to **A**). In the triangle law, **B** starts at the head of **A**, and the resultant **R** closes the triangle. In the parallelogram law, both start at O, and **R** is the diagonal: the same arrow, 6.1 units long, at 25° to **A**. Subtracting **B** means adding −**B**: **A** − **B** is 3.6 units long.*

WORKED EXAMPLE

Walking to a friend

From Bhaktapur Durbar Square, Kiran walks 4 km east and then 3 km north. Find his resultant displacement.

Answer

The two walks are at right angles, so by Pythagoras' theorem

$$R = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5 \text{ km}$$

$$\tan\theta = \frac{3}{4} = 0.75 \quad \Rightarrow \quad \theta = 36.9^\circ$$

His displacement is 5 km, at 36.9° north of east. He has walked a distance of 7 km, but he is only 5 km from where he started.

WORKED EXAMPLE

Crossing the Narayani

A boat that moves at 4 m/s in still water is steered straight across the Narayani, which flows at 3 m/s. Find the boat's resultant velocity. If the river is 200 m wide, how long does the crossing take, and how far downstream does the boat land?

Answer

The boat's own velocity (4 m/s across) and the river's (3 m/s downstream) are at right angles:

$$v = \sqrt{4^2 + 3^2} = 5 \text{ m/s}, \qquad \tan\theta = \frac{3}{4} \Rightarrow \theta = 36.9^\circ$$

The boat moves at 5 m/s, at 36.9° downstream from straight across. The crossing depends only on the velocity across the river: time = 200 m ÷ 4 m/s = **50 s**. In those 50 s the river carries the boat $3 \times 50 =$ **150 m downstream**. (Its path is 250 m long: 5 m/s for 50 s.)

DID YOU KNOW?

If you study Optional Mathematics

For two vectors A and B with an angle θ between them, the length of the resultant can be calculated without a drawing:

$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

At $\theta = 0°$, $\cos\theta = 1$ and R = A + B; at 90°, $\cos\theta = 0$ and it becomes Pythagoras' theorem; at 180°, $\cos\theta = -1$ and R = A − B. For the activity's forces, $R = \sqrt{900 + 1600 + 2 \times 30 \times 40 \times 0.5} = \sqrt{3700} = 60.8$ N, which your drawing should agree with. The rules for adding vectors, in components and coordinates, are in Optional Mathematics: Triangle law of vector addition.

Check 2

Two forces of 6 N and 4 N act on a box. Which of these can **not** be their resultant, whatever the angle between them?

a 2 N **b** 5 N
c 10 N **d** 12 N

Check 3

A boat is steered straight across a river at 3 m/s while the river flows at 4 m/s. What is the boat's speed relative to the bank?

a 1 m/s
b 5 m/s
c 7 m/s
d 12 m/s

A STORY FROM SCIENCE

Newton's parallelogram

The rule for adding forces by a parallelogram is much older than the word "vector". The Flemish engineer Simon Stevin worked out how forces balance on a slope in 1586, with a famous drawing of a loop of balls hanging over a triangular block. Isaac Newton put the parallelogram rule near the start of his *Principia* in 1687, as the first corollary to his laws of motion: a body pushed by two forces at once moves along the diagonal of the parallelogram whose sides the two forces would each have moved it along. The words *vector* and *scalar* came later, in 1846, from the Irish mathematician William Rowan Hamilton; *vector* is Latin for "carrier".

WORKED EXAMPLE

A bus turns a corner

A bus moving east at 10 m/s turns a corner and moves north at 10 m/s. Its speed hasn't changed. Has its velocity changed? Find the change in velocity.

Answer

Change in velocity $= \vec{v}_2 - \vec{v}_1 =$ (10 m/s north) + (10 m/s **west**), since −(10 m/s east) is 10 m/s west. The two are at right angles:

$$\Delta v = \sqrt{10^2 + 10^2} = 14.1 \text{ m/s}$$

with $\tan\theta = 10/10 = 1$, so $\theta = 45^\circ$.

The velocity has changed by 14.1 m/s toward the north-west, even though the speed stayed 10 m/s. A change of direction is a change of velocity, which is why a force is needed to turn a bus round a corner.

WORKED EXAMPLE

Rain on a cyclist

Rain falls straight down at 6 m/s. Dolma cycles east at 4.5 m/s. How does the rain seem to fall to her, and which way should she tilt her umbrella?

Answer

The velocity of the rain **relative to Dolma** is the rain's velocity minus hers: (6 m/s down) + (4.5 m/s **west**). These are at right angles:

$$v = \sqrt{6^2 + 4.5^2} = 7.5 \text{ m/s}, \qquad \tan\theta = \frac{4.5}{6} = 0.75 \Rightarrow \theta = 36.9^\circ$$

To her, **the rain falls at 7.5 m/s, slanting toward her from in front, at 36.9° to the vertical**. She should tilt her umbrella forward by about 37°. This is why rain hits the front window of a moving bus and not the back one.

Check 4

A ball moving at 5 m/s hits a wall head-on and bounces straight back at 5 m/s. What is the change in its velocity?

a zero, because its speed is the same
b 5 m/s
c 10 m/s, away from the wall
d 25 m/s

11.2 Velocity–time graph

A velocity–time graph tells the whole story of a journey in a straight line: when the body sped up, how quickly, when it cruised, when it braked, and how far it went. You met its shapes in Grade 9 Science: a flat line is uniform velocity, a straight sloping line is uniform acceleration, and a curve is changing acceleration. This section is about **drawing** one from measurements and **reading numbers** from it.

Plotting a velocity–time graph

To draw a velocity–time graph from a table of readings:

1. Put **time** on the x-axis (across) and **velocity** on the y-axis (up), because time is the quantity we choose and velocity the one we measure.
2. Choose a **scale** for each axis that makes the graph fill most of the paper, with easy steps: 1 cm for 5 s, 1 cm for 2 m/s, never awkward steps such as 1 cm for 3 s.
3. Label each axis with the quantity and its unit, "time (s)" and "velocity (m/s)", and number the big squares.
4. Plot each reading as a small cross or dot.

5. Join the points. Where the velocity changes steadily, the points lie on straight lines; join them with a ruler. Where it doesn't, draw a smooth curve through them.

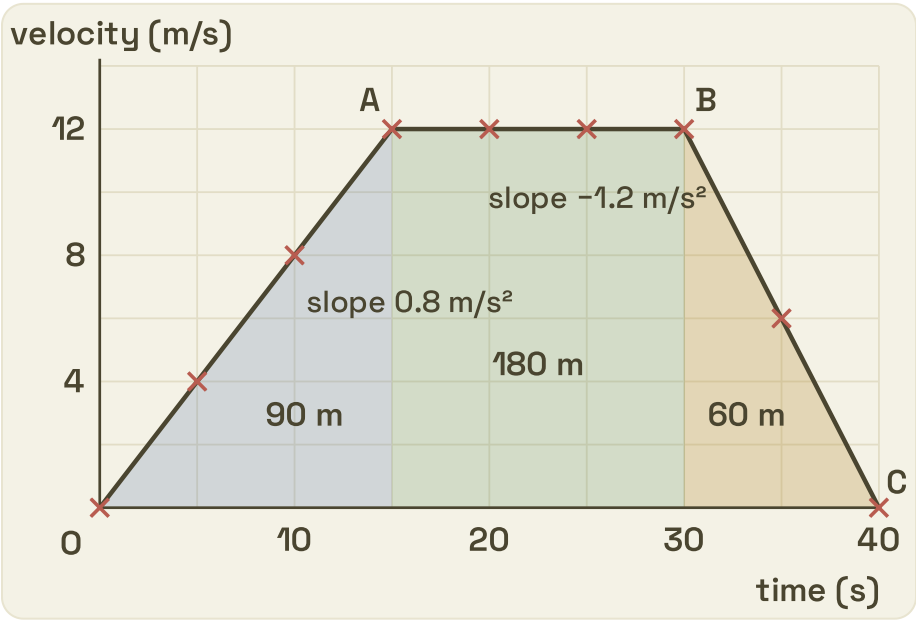


Figure 11.3 *The microbus's velocity–time graph from Practical 22, with the area under each part shaded. The slopes give the accelerations; the areas, 90 m, 180 m and 60 m, add up to the 330 m the microbus traveled.*

Acceleration from the slope

Acceleration is the rate of change of velocity, and on a velocity–time graph that is the **slope** of the line. Pick two points on a straight part, (t_1, v_1) and (t_2, v_2):

$$a = \frac{\text{change in velocity}}{\text{time taken}} = \frac{v_2 - v_1}{t_2 - t_1}$$

- A line sloping **up** has a positive slope: the body is speeding up.
- A **flat** line has zero slope: no acceleration, uniform velocity.
- A line sloping **down** has a negative slope: the body is slowing down, a **retardation** (deceleration).
- The **steeper** the line, the greater the acceleration.

On a curved line the slope is different at every point, and its slope at one moment is the slope of the straight line that just touches the curve there, its **tangent**.

Distance from the area

The **area under a velocity–time graph**, between the line and the time axis, is the **distance** traveled (strictly, the displacement). Why? A bus moving at a steady 10 m/s for 5 s travels $10 \times 5 = 50$ m, and on its graph that is a rectangle 10 high and 5 wide, of area 50. Velocity × time is distance, and height × width is area.

Break the area into shapes you can work out:

Shape under the line	When it appears	Area
rectangle	uniform velocity	width × height
triangle	uniform acceleration from rest, or to rest	$\frac{1}{2}$ × base × height
trapezium	uniform acceleration between two velocities	$\frac{1}{2}$ × (sum of the parallel sides) × width

In working out an area, read the time in seconds and the velocity in m/s from the axes, so that the area comes out in meters.

WORKED EXAMPLE

A bus brakes

A bus moving at 15 m/s brakes uniformly and stops in 5 s. Draw its velocity–time graph, and use it to find its retardation and the distance it travels while braking.

Answer

The graph is a straight line from 15 m/s at 0 s down to 0 m/s at 5 s.

Slope: $a = \dfrac{0 - 15}{5 - 0} = -3$ m/s^2, a **retardation of 3 m/s²**.

Area: a triangle of base 5 s and height 15 m/s: $\frac{1}{2} \times 5 \times 15 = 37.5$ m. **The bus travels 37.5 m while braking.**

Average velocity from the graph

Average velocity is the total displacement divided by the total time taken:

$$\text{average velocity} = \frac{\text{total displacement}}{\text{total time}} = \frac{\text{total area under the graph}}{\text{total time}}$$

So find the whole area first, then divide by the time on the graph's last point.

WORKED EXAMPLE

An elevator in a tall building

An elevator starts from rest and speeds up uniformly to 2 m/s in 2 s, moves up at 2 m/s for 10 s, then slows uniformly to rest in another 2 s. Find how high it rises, and its average velocity.

Answer

The graph is a trapezium: up from 0 to 2 m/s in 2 s, flat at 2 m/s until 12 s, down to 0 at 14 s.

Area = triangle + rectangle + triangle = $\frac{1}{2} \times 2 \times 2 + 10 \times 2 + \frac{1}{2} \times 2 \times 2 = 2 + 20 + 2 =$ **24 m**. (As one trapezium: $\frac{1}{2} \times (14 + 10) \times 2 = 24$ m.)

Average velocity = 24 m ÷ 14 s = **1.71 m/s, upward**. That is about eight stories in 14 s.

CAREFUL

Not the average of the velocities

For the microbus of Practical 22, the average of the nine velocities in the table is 66 ÷ 9 = 7.3 m/s, but its real average velocity is 330 ÷ 40 = 8.25 m/s. Averaging the readings gives the right answer only by luck. The formula $\frac{u + v}{2}$ is right only for **one** stretch of **uniform** acceleration, where it is the height of the trapezium's middle. For a whole journey, always divide the total area by the total time.

Check 5

On a velocity–time graph, a straight line falls from 20 m/s at 0 s to 0 m/s at 4 s. What distance does the body travel?

a 5 m **b** 40 m

c 80 m **d** 160 m

The equations of motion from the graph

The three equations of motion you used in Grade 9 Science can all be read from one velocity–time graph. Take a body moving with uniform acceleration a: its velocity is u at time 0 and v at time t, and it travels a distance s. Its graph is a straight line from $(0, u)$ to (t, v).

1. From the slope: the acceleration is the slope of the line,

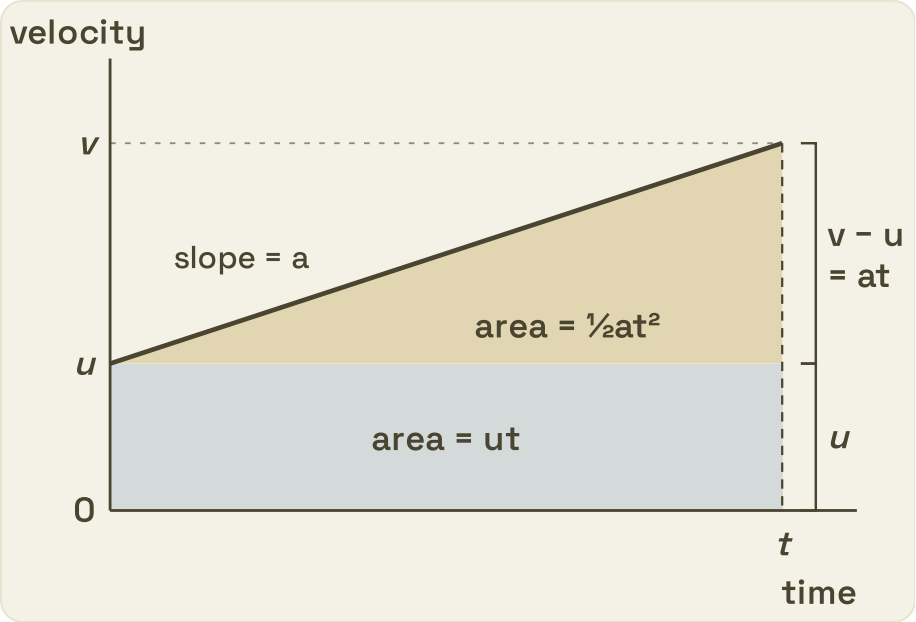


Figure 11.4 *A body speeding up uniformly from u to v in time t. The slope of the line is the acceleration a. The area under it, a rectangle (ut) and a triangle (½at²), is the distance s.*

2. From the area: the area under the line is a trapezium with parallel sides u and v and width t, so

$$s = \frac{(u+v)}{2} \times t \tag{ii}$$

Or split it into a rectangle of height u and a triangle on top, whose height is $v - u = at$:

$$s = ut + \frac{1}{2} \times t \times at \quad \Rightarrow \quad s = ut + \frac{1}{2}at^2 \tag{iii}$$

3. Without the time: from (i), $t = \dfrac{v-u}{a}$. Put this into (ii):

$$s = \frac{(v+u)}{2} \times \frac{(v-u)}{a} = \frac{v^2 - u^2}{2a} \quad \Rightarrow \quad v^2 = u^2 + 2as \tag{iv}$$

When the velocity turns negative

Throw a ball straight up at 20 m/s. Gravity slows it by about 10 m/s every second (taking g ≈ 10 m/s² to keep the numbers simple), so it stops at the top after 2 s and falls back, reaching your hand after 4 s, moving down at 20 m/s. Taking upward as positive, its graph is one straight line from +20 m/s down to −20 m/s, crossing the time axis at 2 s.

The area **above** the axis (0 to 2 s) is $\frac{1}{2} \times 2 \times 20 = 20$ m: the ball rose 20 m. The area **below** the axis (2 to 4 s) is also 20 m, but counted negative: it came down 20 m. So its **displacement** is $20 - 20 = 0$ (it is back in your hand), while the **distance** it traveled is $20 + 20 = 40$ m. An area below the time axis is a displacement in the negative direction.

Check 6

A body's velocity–time graph is a straight line parallel to the time axis, at 8 m/s, from 0 to 5 s. Which statement is correct?

a its acceleration is 8 m/s^2
b it travels 1.6 m
c it travels 40 m with zero acceleration
d it is at rest

11.3 Turning effect of force

Push a door near its hinges and it hardly moves; push it at the handle with the same force and it swings open easily. A force can do more than push a thing along: it can **turn** it about a fixed point or line, the **pivot** (or fulcrum, or axis of rotation). The door turns about its hinges, a wrench (the spanner, as many in Nepal call it) about the nut, a see-saw about its middle, a steering wheel about its column.

Moment of a force

The turning effect of a force is called its **moment** (or torque).

The moment of a force about a point is the product of the force and the perpendicular distance from the point to the line of action of the force:

$$\text{moment} = \text{force} \times \text{perpendicular distance} \qquad M = F \times d$$

The **line of action** of a force is the straight line along which it acts, through the point where it is applied. The distance d must be measured from the pivot **at right angles** to that line.

The SI unit of moment is the **newton meter (N m)**: a force of 1 N acting 1 m from the pivot, at right angles, has a moment of 1 N m.

A moment turns a body either **clockwise** or **anticlockwise**, and the direction matters as much as the size: two equal moments in opposite directions cancel. So a moment is described by its size and its sense of turning.

The moment of a force can be made bigger in two ways: **a bigger force**, or **a bigger perpendicular distance**. The second is usually easier, which is why the bus driver's pipe worked. That is the whole secret of a wrench.

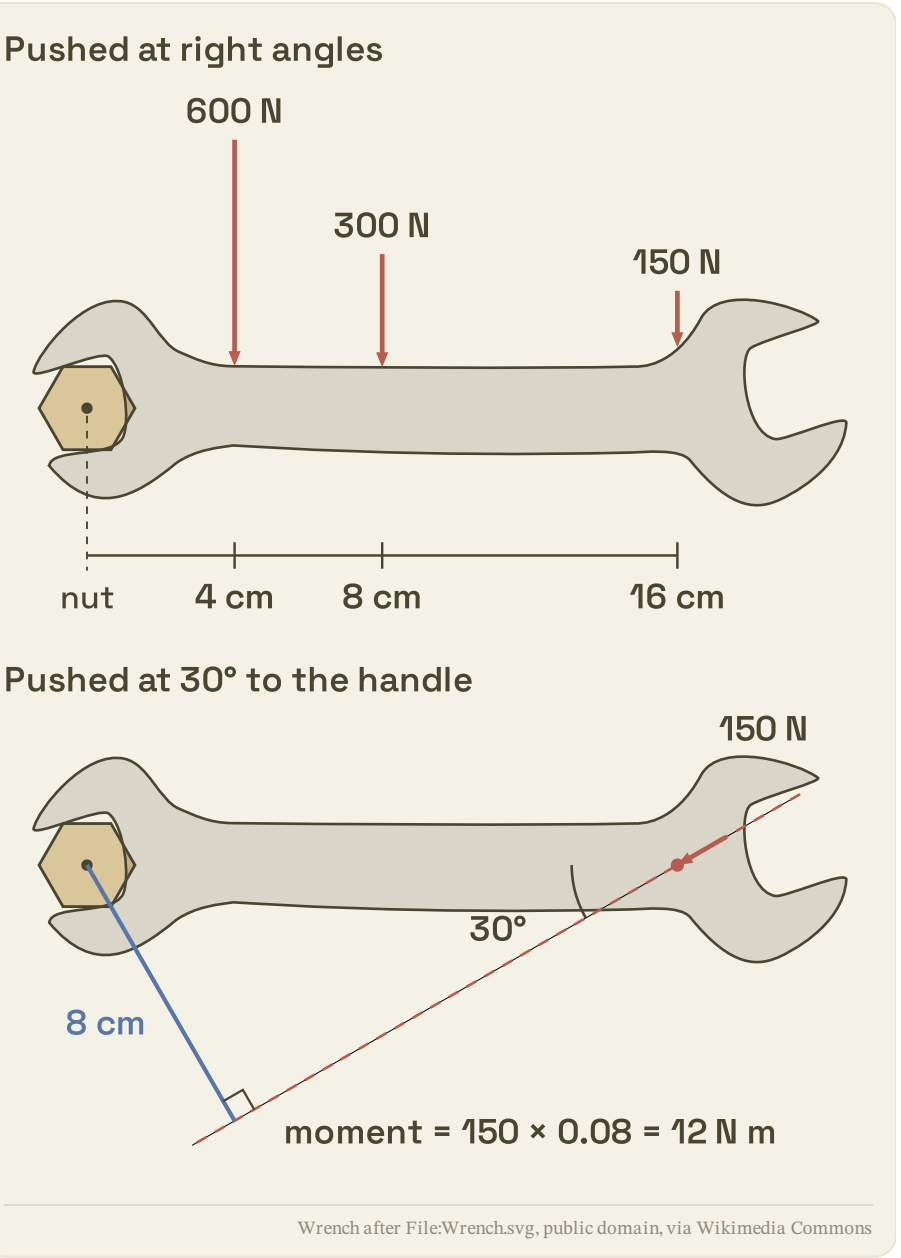


Figure 11.5 *Turning a stiff nut that needs a moment of 24 N m. Gripped 4 cm from the nut, the wrench needs a push of 600 N; at 8 cm, 300 N; at 16 cm, only 150 N. Each push times its distance is 24 N m. Below, the same 150 N pushed at 30° to the handle: the perpendicular distance from the nut to the force's line of action is only 8 cm, so its moment is 12 N m, half as much.*

CAREFUL

A newton meter is not a joule

Work is also force × distance, and 1 N × 1 m is 1 joule. But in work the distance is moved **along** the force, while in a moment it is measured **at right angles** to it, and nothing need move at all: a wrench on a stuck nut has a moment even while the nut refuses to turn. So the moment of a force is always written in **N m**, never in joules.

WORKED EXAMPLE

A wrench on a nut

Pemba pushes on a wrench with a force of 50 N, at right angles to the handle, 30 cm from the center of the nut. What is the moment of his force about the nut?

Answer

F = 50 N, d = 30 cm = 0.30 m

$$M = F \times d = 50 \times 0.30 = 15 \text{ N m}$$

The moment is 15 N m. (Change the distance to meters first, or the answer comes out as 1,500 "N cm".)

WORKED EXAMPLE

Where to push a door

A door needs a moment of 16 N m about its hinges to swing open against its stiff spring. Its handle is 80 cm from the hinges. What force is needed at the handle? What force would be needed 20 cm from the hinges?

Answer

$F = \dfrac{M}{d}$

At the handle: $F = \dfrac{16}{0.80} = 20$ N. **20 N at the handle.**

At 20 cm: $F = \dfrac{16}{0.20} = 80$ N. **80 N, four times as much,** because the distance is a quarter.

Moments around us

- **Door handles** are fixed on the edge farthest from the hinges, for the largest perpendicular distance.

- **Wrenches and lug wrenches** have long handles, and a stuck nut is loosened with a longer one (or a pipe over the handle).
- **A hand pump** at a tube well has a long handle: a small push at its end gives a large moment about the pivot, enough to lift the column of water.
- **Tin snips and garden shears** for cutting thick wire or branches have long handles and short blades.
- **A wheelbarrow** carries its load close to the wheel (the pivot) and is lifted at the far ends of its handles.
- **A bottle opener** grips the cap close to the pivot and is pushed at its far end.
- **Turning a bicycle's pedal** is easiest with the crank horizontal, when the push of your foot is farthest (at right angles) from the axle; with the crank straight down, your push passes through the axle and has no moment at all.

Principle of moments

When a body is balanced and doesn't turn, the moments that try to turn it one way exactly cancel those that try to turn it the other way. In Grade 8 you met this as the principle of the lever (effort × effort distance = load × load distance). In terms of moments:

Principle of moments: when a body is in equilibrium, the sum of the clockwise moments about any point is equal to the sum of the anticlockwise moments about the same point.



Figure 11.6 *A hand pump on a tube well in Nepal. Its long handle turns about a pin near the top of the pump. A push at the handle's end, far from the pin, has a large moment, and the short arm on the other side of the pin lifts the plunger and the water with a large force.*
Photo: Krish Dulal, CC BY-SA 4.0

Couple

Turn a steering wheel and you use two hands: one pushes up on one side of the wheel while the other pulls down on the opposite side. Open a tap, turn a key, unscrew a bottle cap, and your fingers do the same: they push in opposite directions on opposite sides.

A couple is a pair of equal and opposite parallel forces whose lines of action are not the same.

The two forces of a couple add up to zero, so they don't move the body from its place; they only **turn** it. That is the difference from a single force: one push on the edge of a book lying on a table makes it slide as well as turn, but a couple, one finger pushing on each opposite corner, spins it where it lies.

WORKED EXAMPLE

Balancing a see-saw

Aakriti, of mass 30 kg, sits 1.5 m from the middle of a see-saw. Where must her brother Nirajan, of mass 45 kg, sit on the other side to balance it? (Take g = 10 m/s².)

Answer

Aakriti's weight = 30 × 10 = 300 N; Nirajan's = 45 × 10 = 450 N.
By the principle of moments, the anticlockwise moment of one equals the clockwise moment of the other about the pivot:

$$300 \times 1.5 = 450 \times d \quad \Rightarrow \quad d = \frac{450}{450} = 1.0 \text{ m}$$

Nirajan must sit 1.0 m from the middle. The heavier child sits nearer the pivot. (Notice that g cancels: 30 × 1.5 = 45 × d gives the same answer.)

Check 7

A force of 20 N acts at right angles to a wrench, 0.25 m from the nut. What is its moment?

a 0.0125 N m
b 5 J
c 5 N m
d 80 N m

Check 8

Why is it hard to open a door by pushing near its hinges?

a the force is smaller there
b the perpendicular distance from the hinges is small, so the moment is small
c the door is heavier near the hinges
d friction is greater near the hinges

The moment of a couple is one of the forces multiplied by the perpendicular distance between their lines of action, called the **arm of the couple**:

$$\text{moment of a couple} = F \times d$$

Why? Take moments about the middle point between the forces. Each force is $\frac{d}{2}$ from it, and both turn the body the same way, so the total is $F \times \frac{d}{2} + F \times \frac{d}{2} = F \times d$. Take moments about any other point and the answer is the same: about one hand on the wheel, that hand's force has no moment and the other's is $F \times d$. A couple has the same moment about every point.

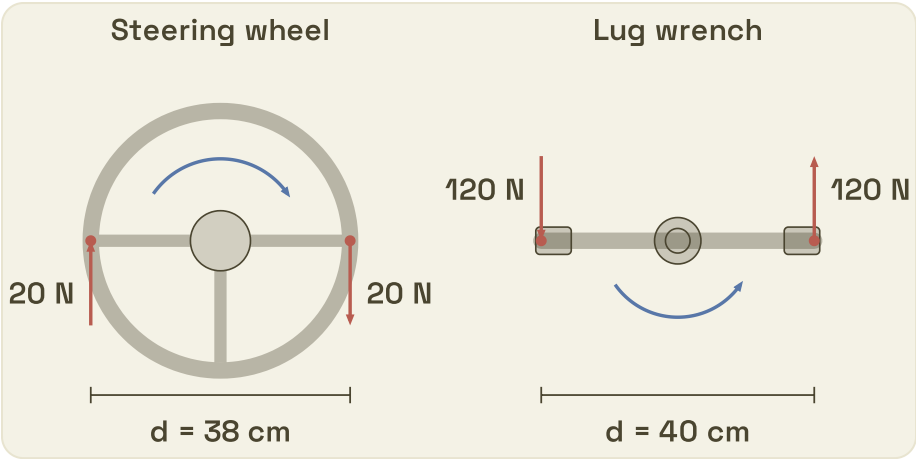


Figure 11.7 *Two couples. Turning a steering wheel 38 cm across, the left hand pushes up and the right hand pulls down, 20 N each: the moment of the couple is $20 \times 0.38 = 7.6$ N m, clockwise, and the car turns right. On a lug wrench, seen end on from where the mechanic stands, one hand pushes down and the other pulls up, 40 cm apart: the couple turns it anticlockwise, which undoes an ordinary nut.*

Couples in daily life

- turning a **steering wheel**, or the handlebar of a bicycle with both hands;
- opening or closing a **water tap**, and turning a **key** in a lock;
- unscrewing the **lid of a jar** or a **bottle cap**;
- turning a **screwdriver**, with the fingers pushing opposite ways on its handle;
- winding a **watch** or a clock;
- a **lug wrench** turned with both hands;
- the forces of the Earth's magnetic field on the two poles of a **compass needle**, which turn it to point north (Unit 14);
- the forces on the two sides of the coil in an **electric motor**, which keep it spinning.



Figure 11.8 *A four-way lug wrench on a car's wheel nut. Each arm ends in a socket for a different size of nut. Pushing down on one arm and pulling up on the opposite one makes a couple: the wrench turns the nut without pushing it sideways.*
Photo: Steffen Heinz Caronna, CC BY-SA 3.0

WORKED EXAMPLE

A couple on a lug wrench

A mechanic holds a lug wrench by the ends of two opposite arms, 40 cm apart, and pushes down with one hand and pulls up with the other, 120 N each. What is the moment of the couple? Could one hand alone, pushing 120 N at one end, loosen a nut that needs 40 N m?

Answer

Moment of the couple $= F \times d = 120 \times 0.40 = 48$ N m. **48 N m**, enough to loosen the nut.

One hand alone at the end of an arm, 20 cm from the nut: $120 \times 0.20 = 24$ N m, **not enough**. It would also push the wrench sideways against the nut. Using both hands as a couple doubles the turning effect and pushes nothing off center.

CAREFUL

A wrench is not a couple

With one hand on a wrench, there is only one force, so it isn't a couple, even though the nut turns. (The nut pushes back on the wrench, but your hand supplies only one force.) Don't give "a wrench" or "a door" as an example of a couple in an exam: give a steering wheel, a tap or a key.

Moment of a force	Couple
The turning effect of one force about a pivot.	Two equal, opposite, parallel forces not in the same line.
Moment = force × perpendicular distance from the pivot.	Moment = one force × the perpendicular distance between the two forces.
Its size depends on the point it is taken about.	Its moment is the same about every point.
A single force can also move the body along (the pivot holds it).	The forces add up to zero: it only turns the body.
Example: opening a door, a wrench on a nut.	Example: a steering wheel, a tap, a key in a lock.

DID YOU KNOW?

Why a compass turns but doesn't move

Put a compass on a table and it swings round to point north, but it doesn't slide toward the North Pole. The Earth's field pulls the needle's north pole northward and its south pole southward with equal forces: a couple. The couple turns the needle until it lies along the field, where both forces act along the same line and the moment is zero. The two forces still cancel, so the needle stays where it is.

Check 9

Which of these is turned by a couple?

a a door pushed open at its handle
b a see-saw with a child on one end
c a water tap opened with the thumb and fingers
d a wheelbarrow lifted at its handles

Check 10

Each hand turns a steering wheel 40 cm across with a force of 15 N, in opposite directions. What is the moment of the couple?

a 3 N m
b 6 N m
c 12 N m
d 600 N m

11.4 Linear momentum

A bicycle rolling at 5 m/s is easy to stop with your hand on its carrier. A loaded truck rolling at the same 5 m/s would crush anyone who tried. A cricket ball tossed gently is easy to catch; the same ball from a fast bowler stings your hands. How hard a moving body is to stop depends on both its **mass** and its **velocity**. Physics puts the two together in one quantity.

Definition

The linear momentum of a body is the product of its mass and its velocity:

$$p = m \times v$$

where p is the momentum, m the mass in kilograms and v the velocity in meters per second. The SI unit of momentum is the **kilogram meter per second (kg m/s)**.

Momentum is a **vector**, with the same direction as the velocity. On a straight road, choose one direction as positive; a body moving the other way then has a negative velocity and a negative momentum. "Linear" means momentum along a line, to tell it

apart from the momentum of a spinning body. A body at rest has no momentum at all, however heavy it is.

WORKED EXAMPLE

Car, motorbike and ball

Find the momentum of: a. a car of mass 1,200 kg moving at 54 km/h; b. a motorbike and its rider, 150 kg together, at 20 m/s; c. a cricket ball of mass 0.16 kg bowled at 25 m/s.

Answer

a. $54 \text{ km/h} = \dfrac{54 \times 1000}{3600} = 15 \text{ m/s}$, so $p = 1200 \times 15 = \mathbf{18{,}000 \text{ kg m/s}}$.

b. $p = 150 \times 20 = \mathbf{3{,}000 \text{ kg m/s}}$. The motorbike is faster, but the car has six times its momentum.

c. $p = 0.16 \times 25 = \mathbf{4 \text{ kg m/s}}$.

Check 11

A parked truck of mass 5,000 kg and a 60 kg runner moving at 5 m/s: which has more momentum?

a the truck, because it is heavier

b the runner, because the truck isn't moving

c both have the same

d neither has any

Momentum and force: Newton's second law

A force acting on a body changes its velocity, and so changes its momentum. Suppose a force F acts on a body of mass m for a time t, and changes its velocity from u to v.

Initial momentum $= mu$; final momentum $= mv$.

Change in momentum $= mv - mu$.

Rate of change of momentum $= \dfrac{mv - mu}{t}$.

Newton found that the rate at which a body's momentum changes is directly proportional to the force on it. This is **Newton's second law of motion**, in the form he gave it:

Newton's second law of motion: the rate of change of momentum of a body is directly proportional to the force applied on it, and takes place in the direction of the force.

So

$$F \propto \frac{mv - mu}{t} \quad \Rightarrow \quad F \propto \frac{m(v-u)}{t}$$

But $\frac{v-u}{t}$ is the acceleration a, so $F \propto ma$, or $F = k\,ma$. The newton is defined as the force that gives 1 kg an acceleration of 1 m/s², which makes $k = 1$:

$$F = ma \qquad \text{and, the same law,} \qquad F = \frac{mv - mu}{t}$$

The second form says more: a force is needed to change a body's momentum, and **the bigger the force, the faster the momentum changes**. Its units agree, too: 1 N = 1 kg m/s², so N × s = kg m/s.

Multiply both sides by t: $Ft = mv - mu$. The force × the time it acts is called the **impulse**, and it equals the change in momentum. The same change in momentum can come from a big force for a short time or a small force for a long time.

WORKED EXAMPLE

Catching a cricket ball

A cricket ball of mass 0.16 kg arrives at 25 m/s. Find the average force on a fielder's hands if she stops it: a. in 0.01 s, with stiff hands; b. in 0.1 s, drawing her hands back as she catches it.

Answer

The ball's momentum changes from $0.16 \times 25 = 4$ kg m/s to 0, a change of 4 kg m/s.

a. $F = \frac{mv - mu}{t} = \frac{0 - 4}{0.01} = -400$ N. **A force of 400 N** (the minus sign means it acts against the ball's motion).

b. $F = \frac{0 - 4}{0.1} = -40$ N. **Only 40 N.** Making the catch last ten times longer makes the force ten times smaller.

Longer stops, smaller forces

The rule $F = \frac{\text{change in momentum}}{t}$ explains many things we do without thinking:

- A fielder **draws her hands back** while catching a fast ball, to make the stop last longer.
- A high jumper lands on a **thick foam mattress**, and a long jumper in **sand**, so they stop over a longer time.
- We **bend our knees** when we jump down from a wall.
- Glass and eggs are packed in **straw, paper or foam**, so a knock stops them gently.

WORKED EXAMPLE

Speeding up a car

A car of mass 1,000 kg speeds up from 10 m/s to 20 m/s in 5 s. Find the change in its momentum and the force that produced it.

Answer

Change in momentum $= mv - mu = 1000 \times 20 - 1000 \times 10 = 10{,}000$ kg m/s.

$$F = \frac{mv - mu}{t} = \frac{10{,}000}{5} = 2{,}000 \text{ N}$$

The momentum grows by 10,000 kg m/s, and the force is 2,000 N (check: $a = 10 \div 5 = 2$ m/s^2, and $F = ma = 1{,}000 \times 2 = 2{,}000$ N).

WORKED EXAMPLE

A ball bounces off a wall

A ball of mass 0.5 kg hits a wall at 10 m/s and bounces straight back at 8 m/s. It touches the wall for 0.05 s. Find the change in its momentum and the average force of the wall on it.

Answer

Take "away from the wall" as positive: $u = -10$ m/s, $v = +8$ m/s.

Change in momentum $= mv - mu = 0.5 \times 8 - 0.5 \times (-10) = 4 + 5 = 9$ kg m/s, away from the wall.

$$F = \frac{9}{0.05} = 180 \text{ N}$$

The momentum changes by 9 kg m/s, and the wall pushes with 180 N. Forgetting the change of direction would give $0.5 \times (10 - 8) = 1$ kg m/s, far too small: a bounce changes the momentum more than a dead stop does.

- A motorcycle **helmet's foam** and a car's **airbags and crumple zones** make a crash last a few hundredths of a second longer, which cuts the force on the head many times.

Conservation of linear momentum

When two bodies collide, or push each other apart, each one's momentum changes. But what one gains, the other loses.

The law of conservation of linear momentum: when no external force acts on a system of bodies, the total momentum of the system stays the same.

Check 12

Why does a fielder draw her hands back while catching a fast cricket ball?

a to reduce the ball's momentum before it arrives

b to increase the time of the catch, so the force on her hands is smaller

c to increase the change in the ball's momentum

d to make the ball bounce out of her hands

A **system** is the group of bodies we are looking at. The forces they exert on one another are **internal forces**; a push or pull from anything outside the group is an **external force**.

Proving the law from Newton's laws

Take two balls, A of mass m_1 moving at u_1 and B of mass m_2 moving at u_2, in the same direction, with A behind and faster ($u_1 > u_2$). A catches up with B and hits it. They touch for a short time t, and then move on with velocities v_1 and v_2.

During the collision, A pushes B forward with a force F_{BA}, and by **Newton's third law** B pushes A backward with an equal and opposite force, $F_{AB} = -F_{BA}$. The two forces last exactly as long, since they act only while the balls touch.

By **Newton's second law**, each force equals the rate of change of the momentum of the ball it acts on:

$$F_{AB} = \frac{m_1 v_1 - m_1 u_1}{t} \qquad F_{BA} = \frac{m_2 v_2 - m_2 u_2}{t}$$

Since $F_{AB} = -F_{BA}$,

$$\frac{m_1 v_1 - m_1 u_1}{t} = -\frac{m_2 v_2 - m_2 u_2}{t}$$

Multiply both sides by t and rearrange:

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

The left side is the total momentum before the collision, and the right side the total momentum after it. **The total momentum before equals the total momentum after**, which is the law of conservation of linear momentum. It holds whether the balls bounce apart, stick together or break, as long as no outside force (such as friction from the ground) acts while they push on each other.

Daily-life uses of the law

Recoil of a gun

Before a gun is fired, the gun and the bullet are at rest: their total momentum is **zero**. The exploding powder pushes the bullet forward and the gun backward with equal

WORKED EXAMPLE

Two trolleys stick together

A trolley of mass 2 kg moving at 3 m/s runs into a 1 kg trolley at rest, and they stick together. Find their velocity just after the collision.

Answer

Total momentum before $= m_1u_1 + m_2u_2 = 2 \times 3 + 1 \times 0 = 6$ kg m/s.
After, they move together, 3 kg at velocity v: total momentum after $= 3v$.
By the conservation of momentum, $3v = 6$, so $v = 2$ m/s.
They move off together at 2 m/s, in the first trolley's direction.

WORKED EXAMPLE

A head-on crash

A truck of mass 3,000 kg moving east at 10 m/s collides head-on with a car of mass 1,000 kg moving west at 20 m/s. The two lock together. Find their velocity just after the crash.

Answer

Take east as positive: the truck's velocity is +10 m/s, the car's −20 m/s.
Total momentum before $= 3000 \times 10 + 1000 \times (-20) = 30{,}000 - 20{,}000 =$ 10,000 kg m/s, east.
After: $(3000 + 1000)\,v = 10{,}000$, so $v = 2.5$ m/s.
The wreck moves east at 2.5 m/s. The car's velocity changes by 22.5 m/s in a fraction of a second, the truck's by only 7.5 m/s: the people in the lighter vehicle feel three times the change.

forces for the same time. Afterward the total momentum must still be zero, so the gun's backward momentum equals the bullet's forward momentum:

$$0 = m_b v_b + m_g v_g \quad \Rightarrow \quad v_g = -\frac{m_b v_b}{m_g}$$

The gun moves backward, and this backward kick is called its **recoil**. It is much slower than the bullet, because it is much heavier. A shooter holds the gun tight against the shoulder, so that the recoil moves the gun and the shooter's body together, a much larger mass, more slowly.

WORKED EXAMPLE

The recoil of a gun

A gun of mass 4 kg fires a bullet of mass 10 g at 400 m/s. Find the velocity with which the gun recoils.

Answer

$m_b = 10\text{ g} = 0.010\text{ kg}$, $v_b = 400\text{ m/s}$, $m_g = 4\text{ kg}$. The total momentum before firing is 0.

$$0 = 0.010 \times 400 + 4 \times v_g \quad \Rightarrow \quad v_g = -\frac{4}{4} = -1\text{ m/s}$$

The gun recoils at 1 m/s, backward (the minus sign). The bullet and the gun get momenta of 4 kg m/s each, in opposite directions.

Stepping out of a boat

The boatman on Phewa Lake and his boat were at rest, with zero total momentum. When he steps forward onto the bank, his feet push the boat backward. He gains forward momentum, so the boat must gain an equal backward momentum, and it slides away from the bank. Step out of a light boat carelessly, and you can end up in the water, as the boat moves away under your back foot.

WORKED EXAMPLE

Jumping from a boat

Lhamu, of mass 50 kg, jumps from a boat of mass 150 kg onto the bank, with a forward velocity of 1.5 m/s. The boat was at rest. With what velocity does the boat move?

Answer

Total momentum before = 0. After: $50 \times 1.5 + 150 \times v = 0$, so $v = -\frac{75}{150} = -0.5$ m/s.

The boat moves backward at 0.5 m/s, a third of her speed, because it has three times her mass.

Rockets and jets

A rocket burns fuel in a chamber and pushes the hot gas out of its nozzle at a very high speed, more than 2 km/s. The gas carries momentum backward; the rocket gains an equal momentum forward. It doesn't need air to push against, which is why rockets work in space. Jet planes work the same way, throwing a stream of air and burnt fuel backward, and so does a balloon let go with its neck open.

Other examples of the same law: a garden or fire **hose** pushes back on the hands that hold it as water rushes out; a **swimmer** pushes water backward to move forward; when you **walk**, your feet push the Earth backward and you move forward (the Earth moves too, but its mass is so great that its velocity is far too small to notice); an **astronaut** floating outside a space station can move by throwing something away from her.



Figure 11.9 *A Falcon 9 rocket lifting off in Florida in March 2017. Its nine engines throw hot gas downward at about 2.8 km/s, well over two tonnes of it every second. The downward momentum given to the gas each second is the upward force on the rocket: about 7 million newtons at lift-off.*
Photo: SpaceX, CC0

A STORY FROM SCIENCE

Who first saw that momentum is kept?

In 1644 the French philosopher René Descartes wrote that the total "quantity of motion", a body's size times its speed, never changes in the world. He was half right: with speed alone, his rules for collisions failed. In 1668 the Royal Society in London asked for papers on collisions, and over that winter three came in, from John Wallis, Christopher Wren and the Dutch scientist Christiaan Huygens. Between them they showed what really stays the same: mass times velocity, counting motion in opposite directions with opposite signs. Newton built it into his laws of motion in 1687, and the law of conservation of momentum has passed every test since, from colliding billiard balls to the particles smashed together at CERN.

Check 13

A gun fires a bullet forward. Why does the gun move backward?

a the bullet pulls the gun after it

b the total momentum was zero, so the gun gains a backward momentum equal to the bullet's forward momentum

c the gun is lighter than the bullet

d the air in front of the bullet pushes the gun back

Check 14

A trolley of mass 1 kg moving at 4 m/s hits an identical trolley at rest, and they stick together. What is their velocity afterward?

a 1 m/s
b 2 m/s
c 4 m/s
d 8 m/s

IN SHORT

- A **scalar** has magnitude only (mass, time, speed, energy, current); a **vector** has magnitude and direction (displacement, velocity, acceleration, force, weight, momentum), drawn as an arrow to scale.
- Vectors add by the **triangle law** (head to tail) or the **parallelogram law** (the diagonal); at right angles, $R = \sqrt{A^2 + B^2}$ and $\tan\theta = B/A$. To subtract, add the negative: $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$.
- On a **velocity–time graph**, the **slope** is the acceleration, the **area** under the line is the distance, and **average velocity** = total area ÷ total time. The three equations of motion come from its slope and area.
- **Moment of a force** = force × perpendicular distance from the pivot to its line of action, in N m. In equilibrium, clockwise moments = anticlockwise moments (**principle of moments**).
- A **couple** is two equal, opposite, parallel forces not in one line; it only turns a body. Its moment = one force × the distance between them (steering wheel, tap, key).
- **Linear momentum** $p = mv$, in kg m/s, a vector. **Newton's second law:** force = rate of change of momentum, $F = \dfrac{mv - mu}{t} = ma$; impulse $Ft = mv - mu$.
- **Conservation of linear momentum:** with no external force, total momentum stays the same: $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$. It explains recoil, rockets and stepping out of a boat.

Exercise

1. Choose the correct option

a. Which group contains only vector quantities?

a mass, velocity, force
b displacement, acceleration, weight
c speed, time, momentum
d work, force, electric current

b. Two forces of 5 N and 12 N act at right angles on a body. Their resultant is:

- **a** 7 N
- **b** 8.5 N
- **c** 13 N
- **d** 17 N

c. The slope of a velocity–time graph gives:

- **a** the distance traveled
- **b** the displacement
- **c** the acceleration
- **d** the average velocity

d. The SI unit of the moment of a force is:

- **a** N/m
- **b** N m
- **c** J
- **d** kg m/s

e. A couple acting on a body:

- **a** moves it along without turning it
- **b** turns it without moving it from its place
- **c** neither moves nor turns it
- **d** always moves it and turns it

f. The rate of change of momentum of a body is equal to:

- **a** its mass
- **b** its velocity
- **c** the force acting on it
- **d** its kinetic energy

g. The momentum of a 2 kg ball moving at 3 m/s is:

- **a** 1.5 kg m/s
- **b** 5 kg m/s
- **c** 6 kg m/s
- **d** 9 kg m/s

2. Give reasons

a. Distance is a scalar quantity but displacement is a vector quantity.

b. The handle of a door is fixed far from its hinges.

c. A long wrench loosens a tight nut more easily than a short one.

d. A cricket player draws her hands back while catching a fast ball.

e. A gun moves backward when a bullet is fired from it.

f. A boat moves away from the bank when a person steps out of it onto the bank.

g. The area under a velocity–time graph gives the distance traveled.

3. Differentiate between

i. Scalar quantity and vector quantity

ii. Moment of a force and couple

iii. Uniform velocity and uniform acceleration on a velocity–time graph

4. Answer the following questions

a. What is a resultant vector? State the triangle law and the parallelogram law of vector addition.

b. How can you subtract one vector from another? A ball moving at 6 m/s toward a wall bounces back at 6 m/s. Explain why its velocity has changed even though its speed hasn't, and find the change.

c. How are the acceleration, the distance traveled and the average velocity found from a velocity–time graph?

d. Derive $v = u + at$ and $s = ut + \frac{1}{2}at^2$ from a velocity–time graph.

e. Define the moment of a force. On what two things does it depend? Give two daily-life examples where we use a long arm to get a large moment.

f. State the principle of moments. A meter stick is pivoted at its middle. A 3 N weight hangs 20 cm to the left of the pivot. Where must a 2 N weight hang to balance it?

g. What is a couple? Show that the moment of a couple is one of the forces multiplied by the perpendicular distance between them. Give four examples of couples from daily life.

h. Define linear momentum. Show that Newton's second law of motion leads to F = ma.

i. State the law of conservation of linear momentum, and prove it for two colliding bodies.

j. How does a rocket move forward in space, where there is no air to push against?

5. Numerical problems

a. Two forces of 6 N and 8 N act on a body at right angles to each other. Find the magnitude and direction of their resultant.

b. Find the resultant of the same two forces (6 N and 8 N) when they act: (i) in the same direction; (ii) in opposite directions.

c. An aircraft heads north at 240 km/h through still air. A wind starts to blow toward the east at 70 km/h. Find the aircraft's resultant velocity over the ground.

d. A car starts from rest and speeds up uniformly to 20 m/s in 8 s. It moves at 20 m/s for 12 s, then stops uniformly in 4 s. Draw its velocity–time graph and find: (i) its acceleration in the first 8 s; (ii) its retardation in the last 4 s; (iii) the total distance traveled; (iv) its average velocity.

e. A cyclist's velocity falls uniformly from 10 m/s to 4 m/s in 3 s as she climbs a hill. Find her retardation and the distance she covers in those 3 s from the velocity–time graph.

f. The table gives the velocity of a toy car every 2 seconds. Plot its velocity–time graph and find the distance it travels in 10 s and its average velocity.

Time (s)	0	2	4	6	8	10
Velocity (m/s)	0	1	2	3	3	3

g. A force of 40 N is applied at right angles to a wrench, 25 cm from the center of a nut. Find the moment of the force. What force would give the same moment at 10 cm?

h. Sonam, of mass 40 kg, sits 2 m from the pivot of a see-saw. How far from the pivot must Rabina, of mass 32 kg, sit on the other side to balance it?

i. The fingers turning a tap push on its handle with two equal and opposite forces of 25 N, 8 cm apart. Find the moment of the couple.

j. A car of mass 1,500 kg moving at 72 km/h is brought to rest in 4 s. Find the change in its momentum and the average braking force.

k. A toy spring cannon of mass 0.8 kg fires a 20 g ball at 12 m/s. Find the velocity of the cannon's recoil.

l. Bishal, of mass 60 kg, runs at 4 m/s and jumps onto a trolley of mass 40 kg at rest. Find the velocity of Bishal and the trolley together.

m. A 2 kg trolley moving at 3 m/s collides head-on with a 1 kg trolley moving at 2 m/s in the opposite direction. They stick together. Find their common velocity.

n. **Think:** An astronaut of mass 100 kg (with her suit) is floating at rest 10 m from her space station, and her safety line has come loose. She throws a 2 kg tool away from the station at 5 m/s. How fast does she move toward the station, and how long does she take to reach it? Why would throwing nothing and "swimming" with her arms not help?

Found a mistake? Please tell us.