Final Exam Model Paper 3

ANSWER KEY

Full marks 75 · 3 hours

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

A practice paper by Science Playground, made to CDC's Specification Grid 2082, Optional Mathematics, Grades 9–10; not an official CDC or exam board paper.

Group A Objective questions — 11 marks

Question 1 — 11 × 1 = 11

A. (i) 3

B. (i) $\begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix}$

C. (ii) $\operatorname{cosec}^2\theta$

D. (i) 1

E. (ii) 60°

F. (i) a reflection in the line $y = x$

G. (ii) $y = -4$

H. (ii) the perpendicular bisector of AB

I. (i) $\begin{pmatrix} 4 \\ -6 \end{pmatrix}$

J. (i) 20%

K. (i) 5

Group B Within one content area — 40 marks

Question 2 — 4 marks

a. $A \times B$ is the set of all ordered pairs whose first element is from A and second element from B: $A \times B = \{(x, y) : x \in A, y \in B\}$. **[1]**

Marking scheme

- The set of all ordered pairs (x, y) with $x \in A$ and $y \in B$.

b. **R_1 is a function**: each of 1, 2 and 3 has exactly one image (1 and 2 may share the image a). **[2]**
R_2 is not: 1 has two images, a and b, and 3 has none.

Marking scheme

- R_1 is a function, since every element of A has exactly one image.
- R_2 is not, since 1 has two images (and 3 has none).

c. $(-1, 3] = \{x : -1 < x \leq 3, x \in R\}$: an open dot at −1, a filled dot at 3, and the line between them. **[1]**

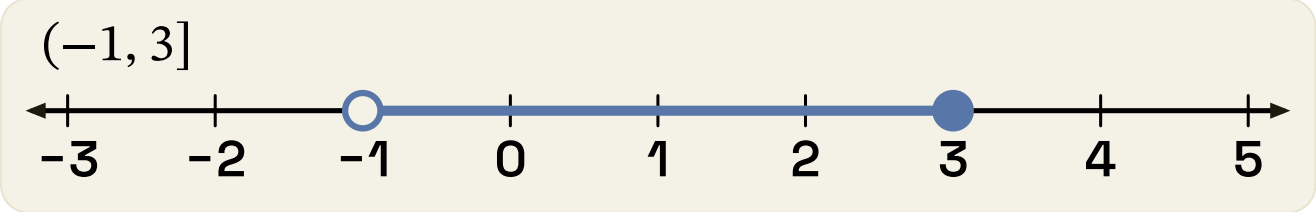


Figure 1 The interval $(-1, 3]$.

Marking scheme

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- An open dot at −1, a filled dot at 3, joined.

Question 3 — 3 marks

a. **A diagonal matrix**: it is square, and every element off the main diagonal is 0. **[1]**

Marking scheme

- Diagonal, since all the elements off the main diagonal are 0.

b. **No.** A scalar matrix is a diagonal matrix whose diagonal elements are all equal, but here they are 2, 5 and −1. **[1]**

Marking scheme

- No, because its diagonal elements 2, 5, −1 are not all equal.

c. **[1]**

$$2A + I = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Marking scheme

- $2A + I = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Question 4 — 3 marks

a. The roots are where the graph cuts the X-axis: $x = 2 + \sqrt{3}$ and $x = 2 - \sqrt{3}$. **[1]**

Marking scheme

- $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

b. **[2]**

$$(2 + \sqrt{3}) + (2 - \sqrt{3}) = 4$$
$$(2 + \sqrt{3})(2 - \sqrt{3}) = 2^2 - (\sqrt{3})^2 = 4 - 3 = 1$$

Sum 4, product 1: both rational, though each root is irrational. (So $2 - \sqrt{3} = \frac{1}{2+\sqrt{3}}$: the two roots are reciprocals.)

Marking scheme

- Sum 4.
- Product $4 - 3 = 1$.

Question 5 — 2 marks

a. $\cos(-60^\circ) = \cos 60^\circ = \frac{1}{2}$ **[1]**

Marking scheme

- $\cos(-60^\circ) = \cos 60^\circ = \frac{1}{2}$.

b. $\sin 330^\circ = \sin(360^\circ - 30^\circ) = -\sin 30^\circ = -\frac{1}{2}$, a negative number. Between 90° and 180° (the second quadrant) every sine is positive. **So there is no such angle.** (The angles with sine $-\frac{1}{2}$ up to 360° are 210° and 330°.) **[1]**

Marking scheme

- No, because $\sin 330^\circ = -\frac{1}{2}$ is negative and the sine is positive in the second quadrant.

Question 6 — 4 marks

a. $\sin^2\theta = 1 - \cos^2\theta$ **[1]**

Marking scheme

- $\sin^2\theta = 1 - \cos^2\theta$.

b. $\sec\theta = \frac{1}{\cos\theta}$ **[1]**

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Marking scheme

- $\sec\theta = \frac{1}{\cos\theta}$.

c. [2]

$$\begin{aligned}\text{LHS} &= (\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta) + (\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta)\\ &= 2\sin^2\theta + 2\cos^2\theta\\ &= 2(\sin^2\theta + \cos^2\theta) = 2 \times 1 = 2 = \text{RHS}\end{aligned}$$

Proved.

Marking scheme

- Both squares expanded, the $2\sin\theta\cos\theta$ terms cancelling.
- $2(\sin^2\theta + \cos^2\theta) = 2$.

Question 7 — 3 marks

a. The kite's height is $\sqrt{50^2 - 40^2} = \sqrt{2500 - 1600} = \sqrt{900} = 30$ m. So $\sin\theta = \frac{30}{50} = \frac{3}{5}$, and [2]

$$\tan\theta = \frac{30}{40} = \frac{3}{4}, \qquad \operatorname{cosec}\theta = \frac{50}{30} = \frac{5}{3}.$$

Marking scheme

- The height 30 m and $\tan\theta = \frac{3}{4}$.
- $\operatorname{cosec}\theta = \frac{5}{3}$.

b. [1]

$$\frac{\frac{3}{5} - \frac{4}{5}}{\frac{3}{5} + \frac{4}{5}} = \frac{-\frac{1}{5}}{\frac{7}{5}} = -\frac{1}{7}$$

Marking scheme

- $-\frac{1}{7}$.

Question 8 — 3 marks

a. $m = -\frac{a}{b}$ (the coefficient of x over the coefficient of y, with the sign changed). [1]

Marking scheme

- $m = -\frac{a}{b}$.

b. [2]

$$\begin{aligned}3x - 4y &= -20\\ \frac{3x}{-20} - \frac{4y}{-20} &= 1\\ \frac{x}{-\frac{20}{3}} + \frac{y}{5} &= 1\end{aligned}$$

x-intercept $-\frac{20}{3}$, y-intercept 5. Check: $y = 0$ gives $3x = -20$; $x = 0$ gives $-4y = -20$, $y = 5$ ✓.

Marking scheme

- $\frac{x}{-\frac{20}{3}} + \frac{y}{5} = 1$.
- x-intercept $-\frac{20}{3}$ and y-intercept 5.

Question 9 — 4 marks

a. Let $P(x, y)$. [2]

$$\begin{aligned}(x-1)^2 + y^2 + (x+1)^2 + y^2 &= 20\\ x^2 - 2x + 1 + x^2 + 2x + 1 + 2y^2 &= 20\\ 2x^2 + 2y^2 &= 18\\ x^2 + y^2 &= 9\end{aligned}$$

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The locus is a circle with center O and radius 3.

Marking scheme

- PA^2 and PB^2 written by the distance formula.
- $x^2 + y^2 = 9$.

b. Externally, m_2 is taken as $-m_2$: **[2]**

$$x = \frac{2 \times 4 - 1 \times 1}{2 - 1} = 7$$
$$y = \frac{2 \times 7 - 1 \times (-2)}{2 - 1} = 16$$

The point is $(7,\ 16)$, beyond D. Check: it is twice as far from C as from D ($6\sqrt{10}$ from C, $3\sqrt{10}$ from D).

Marking scheme

- The external section formula with the values put in.
- $(7,\ 16)$.

Question 10 — 4 marks

a. $P(x,\ y) \to P'(2a - x,\ 2b - y)$ **[1]**

Marking scheme

- $(x,\ y) \to (2a - x,\ 2b - y)$.

b. With $(a,\ b) = (1,\ 2), (x,\ y) \to (2 - x,\ 4 - y)$: $A'(0,\ 0), B'(-2,\ 1), C'(-1,\ -2)$. **[1]**

Marking scheme

- $A'(0,\ 0), B'(-2,\ 1), C'(-1,\ -2)$.

c. **[2]**

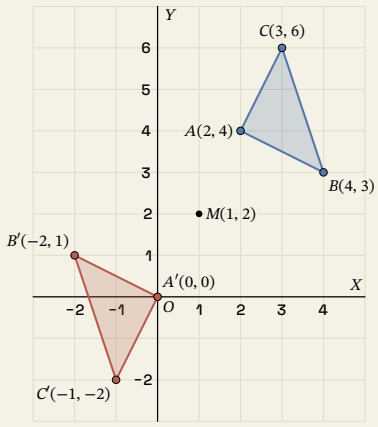


Figure 3 $\triangle ABC$ (blue) and its image $\triangle A'B'C'$ (red) after a half turn about $M(1,\ 2)$.

The center $(1,\ 2)$ is the midpoint of each vertex and its image.

Marking scheme

- $\triangle ABC$ plotted and labelled.
- $\triangle A'B'C'$ plotted and labelled.

Question 11 — 3 marks

a. If two vectors are the two sides of a triangle taken in order, their sum is the third side taken the opposite way: in $\triangle ABC$, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$. **[1]**

Marking scheme

- $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, the third side of the triangle.

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b. [1]

$$\overrightarrow{AC} = \begin{pmatrix} 3+5 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}, \qquad |\overrightarrow{AC}| = \sqrt{64+4} = \sqrt{68} = 2\sqrt{17} \approx 8.25$$

Marking scheme

- $\overrightarrow{AC} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$, magnitude $2\sqrt{17} \approx 8.25$ units.

c. **No.** He walked $|\overrightarrow{AB}| + |\overrightarrow{BC}| = 5 + \sqrt{29} \approx 10.39$ units, but $|\overrightarrow{AC}|$ is the straight distance from A to C, about 8.25. Vectors add head to tail, not by adding their lengths; the two sides of a triangle together are longer than the third. [1]

Marking scheme

- No: the distance walked is $|\overrightarrow{AB}| + |\overrightarrow{BC}|$, longer than the straight line $|\overrightarrow{AC}| \approx 8.25$.

Question 12 — 3 marks

a. The lowest value, the lower quartile Q_1, the median, the upper quartile Q_3 and the highest value. [1]

Marking scheme

- Lowest, Q_1, median, Q_3, highest.

b. $Q_1 = 35, Q_3 = 60$: $\text{Q.D.} = \frac{60-35}{2} = 12.5$ marks. [1]

Marking scheme

- Q.D. = 12.5.

c. **Yes.** The middle half lies inside the box. Section B's box is $56 - 42 = 14$ marks wide; Section A's is $60 - 35 = 25$ marks wide. So Section B's middle half is much more consistent (its Q.D. is 7 against A's 12.5). [1]

Marking scheme

- Yes: B's interquartile range 14 is smaller than A's 25.

Question 13 — 4 marks

a. The sum of an infinite series is the limit of the sequence of its partial sums (the sums of its first 1, 2, 3, … terms), when they head for a number. [1]

Marking scheme

- The limit of its partial sums.

b. **No.** Every partial sum, 0.3, 0.33, 0.333, …, is less than $\frac{1}{3}$, but 0.333… with 3s going on for ever is the sum of the whole infinite series, which means the limit of the partial sums, and that limit is exactly $\frac{1}{3}$. [1]

Marking scheme

- No: 0.333… is the limit of its partial sums, which is exactly $\frac{1}{3}$.

c. $S_1 = 0.3, S_2 = 0.33, S_3 = 0.333, S_4 = 0.3333$. They get closer and closer to $0.333\ldots = \frac{1}{3}$: the sum is $\frac{1}{3}$. [1]

Marking scheme

- 0.3, 0.33, 0.333, 0.3333, heading for $\frac{1}{3}$.

d. **No.** Its partial sums 1, 3, 6, 10, 15, … grow without bound, so they don't head for any number. [1]

Marking scheme

- No, because its partial sums grow without bound.

Group C Across content areas — 24 marks

Question 14 — 6 marks

a. The intercepts are -2 and 4, so in double-intercept form [2]

$$\frac{x}{-2} + \frac{y}{4} = 1 \Rightarrow -2x + y = 4 \Rightarrow y = 2x + 4.$$

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For $P(k,\,2k+4)$: $2\times k+4=2k+4$ ✓, so P is on the road whatever k is.

Marking scheme

- The road $y=2x+4$ (or $2x-y+4=0$).
- P's coordinates satisfy it for every k.

b. $\frac{2k+4}{k}=2+\frac{4}{k}$. As $k\to\infty$, $\frac{4}{k}\to 0$, so the slope heads for **2**, the road's own slope: seen from far along the road, OP runs nearly along it. (For $k=10,\,100,\,1000$: 2.4, 2.04, 2.004.) **[1]**

Marking scheme

- The slope heads for 2.

c. $P(-1,\,2\times(-1)+4)=P(-1,\,2)$, and $OP=\sqrt{(-1)^2+2^2}=\sqrt{5}\approx 2.24$ km. **[1]**

Marking scheme

- $P(-1,\,2)$ and $OP=\sqrt{5}$ km.

d. Put $(0,\,0)$ in $2x-y+4$: $4>0$. So the origin's side is $2x-y+4>0$. For F: $2-3+4=3>0$, so **the farmhouse is on the same side as the origin.** **[2]**

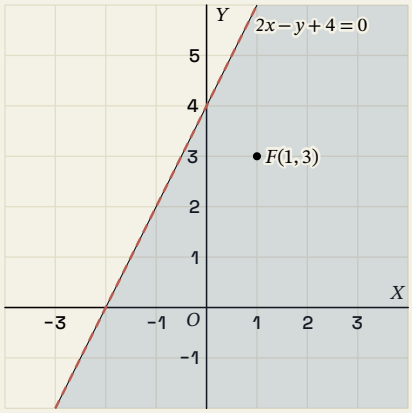


Figure 5 The region $2x-y+4>0$, with the road $y=2x+4$ dashed, and the farmhouse F inside it.

Marking scheme

- $2x-y+4>0$ from testing the origin.
- F gives 3 > 0, so it is on the origin's side, shown on the graph.

Question 15 — 6 marks

a. **[2]**

$$f=\{(1,2),(2,5),(3,10),(4,17),(5,26),(6,37),(7,50)\},\qquad \text{range}=\{2,5,10,17,26,37,50\}.$$

Marking scheme

- f as the seven ordered pairs.
- Its range.

b. **Yes, on** $\{1,\ldots,7\}$: the squares of different positive numbers are different, so no two patterns use the same number of tiles. **With −1 added, no:** $f(-1)=1+1=2=f(1)$, two elements with the same image. **[1]**

Marking scheme

- One-to-one on $\{1,\ldots,7\}$, but not with −1, since $f(-1)=f(1)=2$.

c. In order: 2, 5, 10, 17, 26, 37, 50 ($N=7$). **[3]**

$Q_1=\left(\frac{7+1}{4}\right)^{\text{th}}=2^{\text{nd}}$ item $=5$; $Q_3=\left(\frac{3(7+1)}{4}\right)^{\text{th}}=6^{\text{th}}$ item $=37$.

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$$\text{Q.D.} = \frac{37-5}{2} = 16, \qquad \text{Coefficient of Q.D.} = \frac{37-5}{37+5} = \frac{32}{42} = \frac{16}{21} \approx 0.76$$

Marking scheme

- $Q_1 = 5$ and $Q_3 = 37$.
- Q.D. = 16.
- Coefficient $\frac{16}{21} \approx 0.76$.

Question 16

6 marks

a. **[2]**

$$120^\circ = \left(120 \times \frac{10}{9}\right)^{\text{g}} = \left(\frac{400}{3}\right)^{\text{g}} = 133\tfrac{1}{3}^{\text{g}}, \qquad 120^\circ = \left(120 \times \frac{\pi}{180}\right)^{\text{c}} = \left(\frac{2\pi}{3}\right)^{\text{c}}$$

Marking scheme

- $133\frac{1}{3}$ grades.
- $\frac{2\pi}{3}$ radians.

b. **[2]**

$$l = r\theta = 42 \times \frac{2}{3} \times \frac{22}{7} = 88 \text{ cm}$$

In an hour it turns 360∘, 2π radians: $l = 42 \times 2 \times \frac{22}{7} = 264$ cm (three times 88).

Marking scheme

- 88 cm in 20 minutes.
- 264 cm in an hour.

c. After turning 120∘ from the upright position, the tip's height above the center is $42\cos 120^\circ = 42 \times \left(-\frac{1}{2}\right) = -21$ cm, since $\cos 120^\circ = -\cos 60^\circ$. **Yes, she is right**: the tip is 21 cm below the center (at the 4 o'clock mark). **[1]**

Marking scheme

- Yes: $42\cos 120^\circ = -21$, so the tip is 21 cm below the center.

d. $\frac{l}{\theta} = \frac{r\theta}{\theta} = r = 42$ for every angle other than 0, so as $\theta \to 0$ it heads for **42 cm**, the length of the hand, even though at $\theta = 0$ itself the ratio would be $\frac{0}{0}$. **[1]**

Marking scheme

- 42, since $\frac{l}{\theta} = r$ for every θ ≠ 0.

Question 17

6 marks

a. $\overrightarrow{AA'} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ and $\overrightarrow{BB'} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -\frac{2}{3}\overrightarrow{AA'}$. One is a number times the other, so **yes, they are parallel** (unlike vectors, pointing opposite ways). **[1]**

Marking scheme

- Yes: $\overrightarrow{BB'} = -\frac{2}{3}\overrightarrow{AA'}$.

b. $|\overrightarrow{AA'}| = \sqrt{9+9} = 3\sqrt{2}$, $|\overrightarrow{BB'}| = \sqrt{4+4} = 2\sqrt{2}$, and $\overrightarrow{CC'} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$, so $|\overrightarrow{CC'}| = 2\sqrt{2}$. **[2]**

Marking scheme

- $|\overrightarrow{AA'}| = 3\sqrt{2}$.
- $|\overrightarrow{BB'}| = |\overrightarrow{CC'}| = 2\sqrt{2}$.

c. From $A(1, 4) \to A'(4, 1)$ the coordinates have swapped: try $(x, y) \to (y, x)$. **[3]**
Check: $B(3, 1) \to (1, 3) = B'$ ✓, $C(5, 3) \to (3, 5) = C'$ ✓.
So it is **the reflection in the line** $y = x$. This fits part a: every point moves at right angles to the mirror line, along $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or its opposite, which is why $\overrightarrow{AA'}$ and $\overrightarrow{BB'}$ are parallel.

Marking scheme

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- The rule $(x, y) \to (y, x)$ found from a vertex.
- Checked with the other two vertices.
- Named: reflection in the line $y = x$.

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