Unit 4 · Rational Number

STUDY NOTES

Key terms

Rational number: can be written as a/b, a and b integers, $b \neq 0$; the set **Q** (for **quotient**).

Terminating decimal: stops; the long division reaches remainder 0: $3/8 = 0.375$.

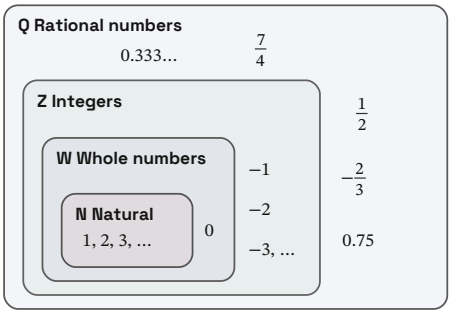
Non-terminating recurring (repeating) decimal: never stops, a digit or block comes round again and again: $4/11 = 0.3636\ldots$.

Additive inverse: of a/b is $-a/b$; the two add to 0.

Multiplicative inverse (reciprocal): of a/b is b/a (turned upside down); the two multiply to 1.

Rational numbers

- integers are closed under $+, -, \times$, **not** $\div$: $7 \div 2 = 3.5$; $3 \div (-4)$ lies between -1 and 0
- "rational" from **ratio**: a ratio of two integers
- every integer is rational: $5 = 5/1$, $-7 = (-7)/1$, $0 = 0/1 \rightarrow Z \subset Q$; not every rational is an integer (1/2)
- $N \subset W \subset Z \subset Q$: each a proper subset of the next (W adds 0, Z the negatives, Q the fractions between)



Each set inside the next.

- bottom never 0 (no number $\times$ 0 makes 6); top can be: $0/5 = 0$
- $-(3/4) = (-3)/4 = 3/(-4)$; $(-3)/(-4) = 3/4$
- one number, many names: $1/2 = 2/4 = (-3)/(-6) = 0.5$

Properties (for + and ×)

Property	Addition	Multiplication
identity	$a + 0 = a$	$a \times 1 = a$
inverse	$\frac{a}{b} + \left(-\frac{a}{b}\right) = 0$	$\frac{a}{b} \times \frac{b}{a} = 1$
commutative	$a + b = b + a$	$a \times b = b \times a$
associative	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
closure	the sum is rational	the product is rational

- **0 has no multiplicative inverse** (1/0 is not a number)
- reciprocal of a negative is negative: $-2/3 \rightarrow$ **$-3/2$** (product $+1$)
- $(3/4) \times (-2/5) = (-2/5) \times (3/4) = -3/10$; its reciprocal $-10/3$
- subtraction and division are **not** commutative: $1/2 - 1/3 = 1/6$ but $1/3 - 1/2 = -1/6$; swapping $\div$ gives the reciprocal
- subtraction closed; division closed except by 0

Fractions into decimals

- divide the top by the bottom; zeros after the point as needed
- $5/8 =$ **0.625** (terminating); 2/3: remainder 2 every time $\rightarrow 0.666\ldots$ (recurring)
- bar over **exactly one block**
  - $0.333\ldots \rightarrow 0.\overline{3}$
  - $0.3636\ldots \rightarrow 0.\overline{36}$
  - $2/7 = 0.\overline{285714}$
- both kinds are **rational**

Which terminate? (lowest terms first)

- denominator with **only 2s and 5s** $\rightarrow$ terminating: 1/2, 1/5, 7/10, 7/25
- **any other prime** (3, 7, 11, ...) $\rightarrow$ recurring: 1/3, 5/7, 1/6
- why: a terminating decimal is tenths, hundredths...: 10, 100, 1,000 = 2s and 5s only
- cancel first: $3/6 = 1/2 = 0.5$; $21/60 = 7/20$ ($2 \times 2 \times 5$) $= 0.35$

- sevenths: $1/7 = 0.\overline{142857}$, 2/7, 3/7, 6/7 the same six digits in a circle; $142857 \times 7 = 999999$

Decimals into fractions

Terminating

1. count the digits after the point
2. digits over 10, 100, 1,000 (a 1 with that many zeros)
3. lowest terms

- $0.75 = 75/100 =$ **3/4**
- $0.125 = 125/1000 =$ **1/8**
- $0.35 = 35/100 =$ **7/20**

Recurring

1. $x =$ the decimal … (i)
2. $\times 10$ (one digit repeats), $\times 100$ (two), $\times 1{,}000$ (three) … (ii)
3. (ii) − (i): the endless tails cancel
4. solve for x

- $x = 0.333\ldots$, $10x = 3.333\ldots \to 9x = 3 \to x = 3/9 =$ **1/3**
- $x = 0.4141\ldots$, $100x = 41.4141\ldots \to 99x = 41 \to$ **41/99**
- $0.666\ldots \to 9x = 6 \to 6/9 =$ **2/3**
- shortcut (block straight after the point): **the block over as many 9s as its digits**: $0.\overline{7} = 7/9$, $0.\overline{123} = 123/999$
- $0.999\ldots =$ **exactly 1** ($9x = 9$; three thirds = 0.999…)

Between two rational numbers

- between 3 and 4: no integer; between **any** two rationals: always another → infinitely many
- **average**: half their sum: between 1/2 and 5/6: ½(1/2 + 5/6) = **2/3**; then ½(1/2 + 2/3) = **7/12**
- or a common denominator with room: 1/2 = 6/12, 5/6 = 10/12 → 7/12, 8/12, 9/12

Kinds of decimal

Decimal	Example	Rational?
terminating	0.375	yes
non-terminating recurring	$0.\overline{36}$	yes
non-terminating, non-recurring	square root of 2 = 1.41421356…	no (Grade 8)

Back to the calculator

- $100 \div 3 = 33.333\ldots = 33 + 1/3 =$ **100/3**: rational; the 3s never stop (3 is not 2 or 5)
- the screen stops only when full; $200 \div 3$ shows 66.666667 (last digit rounded)
- money: at most two decimal places (100 paisa); 10,000 paisa doesn't divide by 3 → Rs 33.33, 33.33, 33.34
- computers work in base 2: 1/10 recurs, so 0.1 + 0.1 + 0.1 == 0.3 is False in Python

COMMON MISTAKES

✗ $6/0 = 0$
✓ no answer: the bottom of a rational number is never 0.

✗ reciprocal of 0 is 0
✓ 0 has no multiplicative inverse.

✗ reciprocal of $-2/3$ is 3/2
✓ keep the sign: $-3/2$.

✗ 3/6 recurs (6 has a 3)
✓ lowest terms first: $1/2 = 0.5$.

✗ bar over two blocks: $0.\overline{3636}$
✓ exactly one block: $0.\overline{36}$.

✗ $0.\overline{41}$: $\times 10$
✓ two digits repeat: $\times 100$, so $99x = 41$.

✗ $0.35 = 35/10$
✓ two digits after the point: $35/100 = 7/20$.

✗ 0.999… is a little less than 1
✓ it is exactly 1.

✗ $1/2 - 1/3 = 1/3 - 1/2$
✓ not commutative: 1/6 and $-1/6$.