Unit 23

Answers: Quantization of Energy

Quick checks

1 b That is postulate (i): in an allowed orbit the electron doesn't radiate. It radiates only when it jumps to a lower orbit, and its angular momentum is always $nh/2\pi$.

2 b $r_n = 0.53\,n^2$ Å, so $r_3 = 9 \times 0.53 = 4.77$ Å, nine times r_1.

3 b $v_n \propto 1/n$, so it slows down; $E_n = -13.6/n^2$ eV, which rises towards 0 as n grows.

4 a $\lambda = h/mv$: at the same speed, the lighter particle has the longer wavelength. The electron is about 1840 times lighter, so its wavelength is about 1840 times longer.

5 b θ is measured from the planes, not from the normal. The angle between the incident and reflected rays is 2θ.

6 b $1/\lambda$, the number of waves in a unit length, is the **wave number**.

7 d $L = nh/2\pi$, and for n = 2, $L = 2h/2\pi = h/\pi$.

8 a $n = I/e = 0.02/1.6 \times 10^{-19} = 1.25 \times 10^{17}$ a second.

9 b Ionization energies are a few eV (13.6 eV for hydrogen); nuclear binding energies are millions of eV (MeV).

10 d For a one-electron ion, $E_n = -13.6\,Z^2/n^2$ eV. With Z = 2, the ground state is -54.4 eV, so 54.4 eV removes the electron. (Removing the first electron from a neutral helium atom takes 24.6 eV; Bohr's model can't calculate that.)

11 a $E_5 = -13.6/5^2 = -0.544$ eV.

12 c $\Delta E = 13.6 \times Z^2 \left(1 - \frac{1}{4}\right) = 13.6 \times 4 \times \frac{3}{4} = 3 \times 13.6 = 40.8$ eV.

13 d $E_3 = -13.6/9 = -1.51$ eV, so 1.51 eV takes it to 0, free.

14 b 4860 Å (486 nm) is visible: H-beta, the jump from n = 4 to n = 2, a Balmer line.

15 a In Bohr's model, $v_n \propto Z/n$: with the same Z, $v_5/v_2 = 2/5$, so $v_5 = 2v/5$. (Bohr's formulas apply only to an atom or ion with one electron.)

16 d The potential energy is $PE_n = 2E_n = -27.2/n^2$ eV: −27.2 eV for n = 1, −6.8 eV for n = 2. The difference is 20.4 eV, so with the ground state's PE as zero, the first excited state's is +20.4 eV. (Only differences of potential energy matter; the zero can be chosen anywhere.)

17 a X-ray wavelengths (about 0.1 nm) are about the spacing of atoms, so crystals diffract them (Bragg's law). Infrared and ultraviolet wavelengths are far too long; γ-rays are mostly too short.

18 b $\lambda_{\min} = \dfrac{12\,400}{V} = \dfrac{12\,400}{20\,000} = 0.62$ Å.

19 a $f_{\max} = eV/h = 1.6 \times 10^{-19} \times 42\,000/6.62 \times 10^{-34} = 1.0 \times 10^{19}$ Hz.

20 b Radiotherapy uses carefully aimed doses of X-rays to kill cancer cells. Finding fractures is diagnosis, not therapy.

21 b Photons of up to 10 keV, $\lambda_{\min} = 1.24$ Å: X-rays.

22 a $d = \dfrac{\lambda}{2\sin\theta} = \dfrac{1}{2 \times 0.174} = 2.88$ Å, nearest 2.8 Å. (The real spacing in rock salt is 2.82 Å.)

18. a. (i) An electron goes round the nucleus only in certain allowed circular orbits, the **stationary orbits**, in which it does **not radiate** energy, so its energy stays constant. (ii) These are the orbits for which its angular momentum is a whole-number multiple of $h/2\pi$: $mvr = nh/2\pi$. (iii) The atom gives out or takes in energy only when the electron jumps between stationary orbits, as one photon: $hf = E_2 - E_1$.

b. **Radius.** The electric attraction provides the centripetal force: $\dfrac{e^2}{4\pi\varepsilon_0 r_n^2} = \dfrac{mv_n^2}{r_n}$.

With $v_n = \dfrac{nh}{2\pi m r_n}$, this gives $r_n = \dfrac{n^2h^2\varepsilon_0}{\pi m e^2} = 0.53\,n^2$ Å.

Speed. $v_n = \dfrac{nh}{2\pi m r_n} = \dfrac{e^2}{2\varepsilon_0 nh} = \dfrac{2.19 \times 10^6}{n}$ m/s.

Energy. $KE = \dfrac{1}{2}mv_n^2 = \dfrac{e^2}{8\pi\varepsilon_0 r_n}$; $PE = -\dfrac{e^2}{4\pi\varepsilon_0 r_n}$; so $E_n = -\dfrac{e^2}{8\pi\varepsilon_0 r_n} = -\dfrac{me^4}{8\varepsilon_0^2 n^2 h^2} = -\dfrac{13.6}{n^2}$ eV. (The full working is in Section 23.2.)

19. **Postulates:** (i) electrons go round the nucleus only in certain stationary orbits, without radiating; (ii) in these orbits the angular momentum is $mvr = nh/2\pi$; (iii) a photon of energy $hf = E_2 - E_1$ is given out (or taken in) when an electron jumps from an orbit of energy E_2 to one of energy E_1 (or back).

The spectrum: the theory gives the energies $E_n = -13.6/n^2$ eV. In hot or excited hydrogen, electrons are raised to higher orbits and then jump down. A jump from n_2 to n_1 gives a photon of wavelength

$$\frac{1}{\lambda} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right), \quad R = \frac{me^4}{8\varepsilon_0^2 ch^3} = 1.097 \times 10^{7-1}$$

Only these wavelengths are possible, so the spectrum is separate lines. Jumps ending on the same orbit form a series: Lyman ($n_1 = 1$, ultraviolet), Balmer (2, visible), Paschen (3), Brackett (4) and Pfund (5, infrared). The wavelengths calculated agree with those measured.

20. The levels are −13.6, −3.4, −1.51 and −0.85 eV for n = 1 to 4. A 12.2 eV electron can give an atom up to 12.2 eV: enough to reach n = 2 (10.2 eV needed) or n = 3 (12.09 eV), but not n = 4 (12.75 eV). So atoms are raised to n = 3 at most, and coming down give three lines:

- 3 → 2: $\frac{1}{\lambda} = R\left(\frac{1}{4} - \frac{1}{9}\right)$, $\lambda = \mathbf{656}$ (red, Balmer);
- 2 → 1: $\frac{1}{\lambda} = R\left(1 - \frac{1}{4}\right)$, $\lambda = \mathbf{121.6}$ (ultraviolet, Lyman);
- 3 → 1: $\frac{1}{\lambda} = R\left(1 - \frac{1}{9}\right)$, $\lambda = \mathbf{102.6}$ (ultraviolet, Lyman).

21.

a. The **excitation potential** is the accelerating voltage that gives an electron enough energy to raise an atom from its ground state to an excited state (for hydrogen, 10.2 V to n = 2). The **ionization potential** is the accelerating voltage that gives an electron enough energy to remove the atom's electron completely from the ground state (13.6 V for hydrogen).

b. $E_n = -13.6/n^2$ eV, so the ground state is $E_1 = -13.6$ eV and a free electron has $E_\infty = 0$. The ionization energy is $E_\infty - E_1 = 13.6$ eV. An electron gains 13.6 eV when accelerated through 13.6 V, so the ionization potential is **13.6 V**.

c. An **emission spectrum** is the light given out by a substance whose atoms or molecules are excited (heated, or by an electric discharge) and fall back to lower levels. It is **continuous** from hot solids, liquids and dense gases, **lines** from separate atoms of a gas (each line one jump, the lines a fingerprint of the element), and **bands** from molecules.

d. At room temperature almost every hydrogen atom is in its **ground state**, n = 1: the energy needed to reach n = 2, 10.2 eV, is hundreds of times the energy of the atoms' random motion, so hardly any are excited. Absorption lifts an electron from the level it is in, so all the absorption starts from n = 1: the jumps 1 → 2, 1 → 3, ..., which are the Lyman series (in the ultraviolet). Balmer absorption would need atoms already in n = 2, and there are almost none.

22. a. X-rays are **electromagnetic waves** of very short wavelength (about 0.06 Å to 100 Å), carrying no charge. They are produced when **fast electrons are suddenly stopped** by a target of a heavy metal: in a Coolidge tube, electrons from a hot filament are accelerated through a high voltage onto a tungsten target, and part of their energy comes out as X-ray photons (Section 23.14).

b. **Penetrating power** (quality) is how easily the X-rays pass through matter; it depends on the energy of each photon. **Intensity** is the energy carried by the X-rays per second through a unit area; it depends on how many photons come out each second.

i. The penetrating power is controlled by the **voltage** across the tube: a higher voltage gives faster electrons, more energetic photons of shorter wavelength ($\lambda_{\min} = hc/eV$), hard X-rays.

ii. The intensity is controlled by the **filament current**: a hotter filament gives out more electrons each second, so more X-ray photons.

c. **No.** Aluminium has a low atomic number (13), so it gives very few X-rays, and a low melting point (660 °C), so the heat of the electron beam would melt it. A target needs a high atomic number and a high melting point, like tungsten.

23. a. A crystal's atoms lie in parallel planes a distance d apart. X-rays falling at a glancing angle θ are partly reflected by each plane. The rays from neighboring planes differ in path by $2d\sin\theta$ (Figure 23.13), so they reinforce, giving a strong reflection, only when

$$2d\sin\theta = n\lambda, \quad n = 1, 2, 3, \ldots$$

This is Bragg's law. Measuring θ for a crystal of known d gives λ (or, for known λ, gives d).

b. $d = 3.08 \times 10^{-10}$ m, θ = 12°, n = 1:

$$\lambda = 2d\sin\theta = 2 \times 3.08 \times 10^{-10} \times 0.208 = \mathbf{1.28 \times 10^{-10}}$$

24. a. $\lambda = \dfrac{h}{p} = \dfrac{h}{\sqrt{2m \cdot KE}}$. For the same kinetic energy, the smaller the mass, the longer the wavelength: the **electron**, about $\sqrt{1836} \approx 43$ times longer than the proton's.

b. Radius $r_1 = 0.53 \times 10^{-10}$ m, so the circumference is

$$2\pi r_1 = 2\pi \times 0.53 \times 10^{-10} = \mathbf{3.33 \times 10^{-10}}$$

Speed $v_1 = 2.19 \times 10^6$ m/s, so the momentum is

$$p = mv_1 = 9.1 \times 10^{-31} \times 2.19 \times 10^6 = \mathbf{1.99 \times 10^{-24}}$$

de Broglie wavelength:

$$\lambda = \frac{h}{p} = \frac{6.62 \times 10^{-34}}{1.99 \times 10^{-24}} = \mathbf{3.33 \times 10^{-10}}$$

The wavelength **equals the circumference** of the first orbit: exactly one wavelength fits round it, as $2\pi r = n\lambda$ with n = 1 requires (Example 23.17).

25. a. From the **kinetic energy of the electrons**. When they are stopped in the target, only about 0.5–1 % of their energy becomes X-rays; the rest goes into shaking the target's atoms, that is, heat.

b. The heat is the other 99.5 % of the electrons' power, so the total power is

$$P = \frac{720}{0.995} = 723.6$$

and the current is

$$I = \frac{P}{V} = \frac{723.6}{10\,000} = 0.0724 = \mathbf{72.4}$$

The speed, from $\frac{1}{2}mv^2 = eV$:

$$v = \sqrt{2\frac{e}{m}V} = \sqrt{2 \times 1.8 \times 10^{11} \times 10^4} = \sqrt{3.6 \times 10^{15}} = \mathbf{6.0 \times 10^7}$$

26. a. The ground state is at -21.8×10^{-19} J, so a photon of 21.8×10^{-19} J (13.6 eV) brings the electron exactly to zero energy: it is **removed from the atom** (the atom is ionized), with practically no kinetic energy left.

b. i. The electron goes from a level of higher energy (-2.4×10^{-19} J) to one of lower energy (-5.4×10^{-19} J). The energy it loses can't vanish: it is carried away by **one photon** whose energy is the difference between the two levels.

ii. $\Delta E = -2.4 - (-5.4) = 3.0 \times 10^{-19}$ J, so

$$\lambda = \frac{hc}{\Delta E} = \frac{6.63 \times 10^{-34} \times 3 \times 10^8}{3.0 \times 10^{-19}} = \mathbf{6.6 \times 10^{-7}}$$

the red Balmer line (656 nm).

27. a. The **wave number** is the number of waves in a unit length (one meter), the reciprocal of the wavelength: $\bar{\nu} = 1/\lambda$, in m^{-1}.

b. i. −0.85 eV is n = 4. From n = 4 the possible jumps are 4 → 3, 4 → 2, 4 → 1, 3 → 2, 3 → 1, 2 → 1: $\dfrac{4 \times 3}{2} = \mathbf{6\ different\ wavelengths}$.

ii. 1. The most energetic photon comes from the largest jump, −0.85 → −13.6 eV: 12.75 eV. By Einstein's photoelectric equation,

$$KE_{\max} = hf - \phi = 12.75 - 5.6 = \mathbf{7.15} \approx 7.2$$

(Only three of the six photons, those of 4 → 1, 3 → 1 and 2 → 1, have more than 5.6 eV; the others give no photoelectrons at all.)

2. The wavelength of that photon is $\lambda = \dfrac{hc}{E} = \dfrac{6.63 \times 10^{-34} \times 3 \times 10^8}{12.75 \times 1.6 \times 10^{-19}} = \mathbf{9.75 \times 10^{-8}\ m}$, in the ultraviolet.

28. a. (i) **Electron diffraction**: electrons passed through a thin crystal (or a pair of slits) make a diffraction pattern, as waves do. (ii) The **photoelectric effect**: light knocks out electrons in packets of energy hf, with no delay, whatever its intensity.

b. i. The shortest wavelength comes from the largest energy jump: from **−0.54 eV to −3.40 eV** (the top level to the bottom).

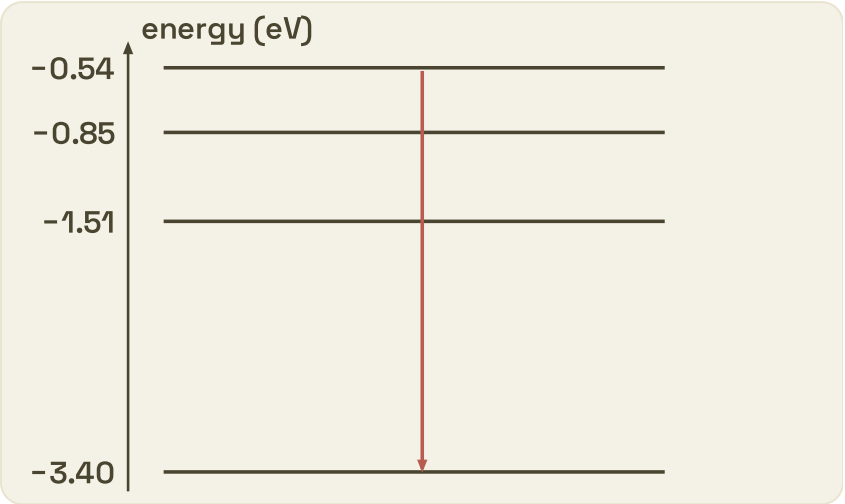


Figure 23.17 *The jump from −0.54 eV to −3.40 eV, the largest of the four levels, gives the shortest wavelength.*

ii. $\Delta E = -0.54 - (-3.40) = 2.86 \text{ eV} = 2.86 \times 1.6 \times 10^{-19} = 4.58 \times 10^{-19} \text{ J}$.

$$\lambda = \frac{hc}{\Delta E} = \frac{6.63 \times 10^{-34} \times 3 \times 10^{8}}{4.58 \times 10^{-19}} = \mathbf{4.35 \times 10^{-7}}$$

c. i. The **de Broglie wavelength** is the wavelength of the wave that goes with a moving particle: $\lambda = h/p = h/mv$.

ii.
$$v = \frac{h}{m\lambda} = \frac{6.63 \times 10^{-34}}{9.1 \times 10^{-31} \times 4.35 \times 10^{-7}} = \mathbf{1.67 \times 10^{3}}$$

A slow electron, about 6000 km/h.