Unit 19

Alternating Current

This unit online, with its interactives:
science-playground.com/s/12f19

Put your ear near the gray transformer box on an electric pole in your tole, and on a quiet night you will hear it hum, a low, steady note. That hum is the sound of the current in your home's wires turning round, stopping and running the other way, a hundred times every second. The electricity in a socket is not a steady flow like a battery's; it is **alternating current**, and the whole world's power grid runs on it.

This unit is about how such a current behaves. In a plain resistor it does what a battery's current would. But in a coil it falls behind the voltage, in a capacitor it runs ahead, and in a circuit with both it can grow huge at one exact frequency: the trick a radio uses to pick one station out of hundreds. You will meet the root-mean-square value that makes "230 V" mean something, the rotating arrows (phasors) that make the algebra easy, impedance, resonance, the quality factor, and why a current can flow without using any power at all.

A STORY FROM SCIENCE

Nikola Tesla and the war of the currents

Nikola Tesla (1856–1943) was born to Serbian parents in Smiljan, a village in today's Croatia. As a young engineer he worked out the **rotating magnetic field**: three alternating currents, out of step with one another, flowing round three coils make a magnetic field that turns by itself, and a motor placed in it turns too, with no brushes or sparking contacts. It is still how almost every big motor works, from water pumps to the fans in a factory. He also designed the **three-phase system** for sending power over long distances.
Tesla moved to the United States in 1884, worked briefly for Thomas Edison, and then went his own way. In 1888 he sold the rights to his a.c. motors, dynamos and transformers to the industrialist **George Westinghouse**. Edison's company was building **direct-current** (d.c.) power stations, and the late 1880s became a bitter fight, the "war of the currents", over which kind the world would use.
A.c. won because of the transformer: its voltage can be stepped **up** for sending over long lines with little loss, then stepped **down** again for safe use in homes. Westinghouse lit the World's Fair in Chicago in 1893 with Tesla's system, and won the contract for the first big power station at **Niagara Falls**, which began sending power to the city of Buffalo, about 40 km away, in 1896. In 1891 Tesla also invented the **Tesla coil**, a resonant transformer that makes very high a.c. voltages at high frequency; its ideas went into early radio. The SI unit of magnetic flux density, the tesla (T), is named after him.

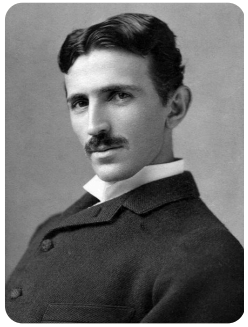


Figure 19.1 *Nikola Tesla in about 1890, when his a.c. motors and transformers were being built by Westinghouse.*
Photo: Napoleon Sarony, Public domain

DID YOU KNOW?

Nepal's first power station

Nepal's first hydropower station, at **Pharping** south of Kathmandu, began generating in 1911 with 500 kW, only fifteen years after Niagara. It made alternating current, as every station in Nepal still does: the national grid runs at **50 Hz** (cycles a second), and the socket in your home is rated at about 230 V.



Figure 19.2 *The powerhouse of the Pharping hydropower station (1911), Nepal's first.*
Photo: Ganesh Paudel, CC BY-SA 4.0

DID YOU KNOW?

The metal detector at the airport

A walk-through metal detector at Tribhuvan Airport has large coils of wire in its frame, carrying a rapidly changing current. When you walk through with metal on you, the coils' changing magnetic field drives tiny **eddy currents** in the metal, and their magnetic field changes the current in the detector's coils by a very small amount. The circuit is built to notice that tiny change, so it must be extremely sensitive: a small key or a belt buckle is enough. How a coil responds to a changing current, its inductance, is exactly what this unit is about.

21.

a. Define resonance in an a.c. circuit, with one example.

b. Prove that the resonant frequency of an a.c. series LCR circuit is $f = \dfrac{1}{2\pi\sqrt{LC}}$.

c. The maximum capacitance of a variable capacitor is 33 pF. What self-inductance must be connected to it for the natural frequency of the LC circuit to be 810 kHz, a frequency in Radio Nepal's medium-wave (AM) band?

A companion to your textbook, not a replacement for it: study from the book your school uses, and where the two differ, follow your teacher. Found a mistake? Please tell us.

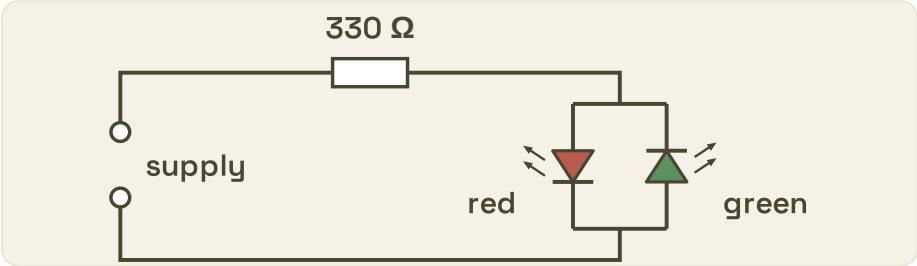


Figure 19.3 *The circuit for Activity 19.1: two LEDs side by side, opposite ways round, in series with a 330 Ω resistor. On a battery only one of them can conduct; on a.c. each conducts in turn, for its own half of every cycle.*

19.1 Introduction

Most of the electricity we use comes from **generators**: coils of wire turned in a magnetic field (or magnets turned inside coils) by water, steam or wind. As a coil turns steadily, the magnetic flux through it rises and falls, and the e.m.f. induced in it follows a sine curve. So a generator's e.m.f., and the current it drives, vary **sinusoidally** with time:

$$V = V_0 \sin \omega t \quad \text{or} \quad V = V_0 \cos \omega t$$
$$I = I_0 \sin \omega t \quad \text{or} \quad I = I_0 \cos \omega t$$

- V is the **instantaneous** e.m.f. or potential difference, its value at time t;
- V_0 is its greatest value, the **peak value** or **voltage amplitude**;
- ω is the **angular frequency**, in rad/s: $\omega = 2\pi f$, where f is the frequency in hertz (Hz), the number of complete cycles in a second.

Whether you write sin or cos only decides where you start the clock: a cosine is a sine that started a quarter of a cycle earlier.

For example, if $V = 10 \sin 40\pi t$ volts, then $V_0 = 10$ V and $\omega = 40\pi$ rad/s, so $f = \dfrac{\omega}{2\pi} = 20$ Hz.

Why does the world use a.c.? Mainly because a **transformer** can change an a.c. voltage up or down, and a transformer works only with a changing current. Power is sent across the country at very high voltage (132 kV and more on Nepal's main lines), which means a small current, so little energy is lost as heat in the long wires (I^2R); near your home it is stepped down to a safe 230 V. With d.c. there was no simple way to do this until modern electronics.

Check 1

A generator's e.m.f. is $V = 325 \sin 100\pi t$ volts. Its frequency is:

a 100 Hz
b 50 Hz
c 100π Hz
d 325 Hz

19.2 Alternating current and e.m.f.

A current or e.m.f. whose **size changes continuously with time and whose direction reverses periodically** is called an **alternating current** or **alternating e.m.f.** (a.c.). Its instantaneous values are

$$I = I_0 \sin \omega t \qquad \text{and} \qquad V = V_0 \sin \omega t$$

where I_0 is the **peak value of the current** (the current amplitude) and V_0 the **peak value of the e.m.f.** (the voltage amplitude).

$$\omega = 2\pi f, \qquad T = \frac{1}{f} = \frac{2\pi}{\omega}$$

where T is the **period**, the time for one complete cycle. In one cycle the current rises from zero to $+I_0$, falls through zero to $-I_0$ (the minus sign means it flows the other way), and comes back to zero.

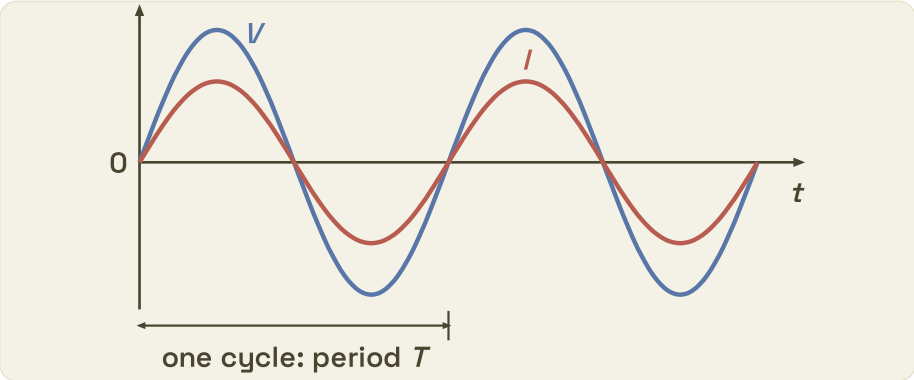


Figure 19.4 *An alternating e.m.f. $V = V_0 \sin \omega t$ (blue) and the current it drives through a resistor, $I = I_0 \sin \omega t$ (red), over two cycles. Both rise, fall, reverse and return together. One cycle takes the period T; at 50 Hz, T = 0.02 s.*

Check 2

Nepal's mains supply has a frequency of 50 Hz. How many times does the current change direction in one second?

a 25
b 50
c 100
d 314

19.3 Root mean square (rms) value of a.c.

A meter on an a.c. supply shows one steady number, "230 V", though the voltage is changing all the time. What does that number mean?

It is defined by **heating**, the thing every current does in a resistance:

The **root mean square (rms) value** of an alternating current is the **steady (d.c.) current that, flowing through the same resistance for the same time, produces the same amount of heat** as the alternating current does.

So the rms value is the a.c.'s **d.c. equivalent** for heating. It is written I_{rms}, or I_v (the **virtual** value), or I_{eff} (the effective value). Every a.c. meter, and every label on an appliance, gives rms values unless it says otherwise.

Relation between the rms value and the peak value

Let the current through a resistance R be $I = I_0 \sin \omega t$.

1. **Try the plain average first.** Over a whole cycle, $\sin \omega t$ is positive for half the time and negative, by the same amounts, for the other half. So the average current is

$$\langle I \rangle = I_0 \langle \sin \omega t \rangle = 0$$

(the angle brackets ⟨ ⟩ mean "the average over a cycle"). But the resistor certainly gets hot, so the plain average tells us nothing about the heating.

19. a. Define the impedance of an LCR circuit.

b. The series LCR circuit above is connected across an a.c. source of e.m.f. 50 V and 50 Hz. i. Calculate the impedance of the circuit, if L = 60 mH, C = 0.50 μF and R = 300 Ω. ii. How does resonance occur in this circuit? Write the expression needed.

c. A device X is connected across an a.c. source of voltage $V = V_0 \sin \omega t$, and the current through X is $I = I_0 \sin\left(\omega t + \frac{\pi}{2}\right)$. i. Identify the device X, and write an expression for its reactance. ii. Draw a graph of the voltage and the current over one cycle for X.

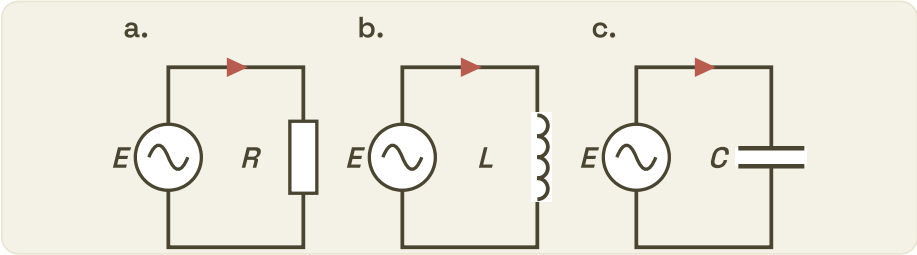


Figure 19.25 *For question 20: three circuits with the same current. a. A resistor R across an a.c. source E. b. An inductor L across a source E. c. A capacitor C across a source E.*

the bulb glows with some brightness. How will its glow change if a capacitor is added in series?

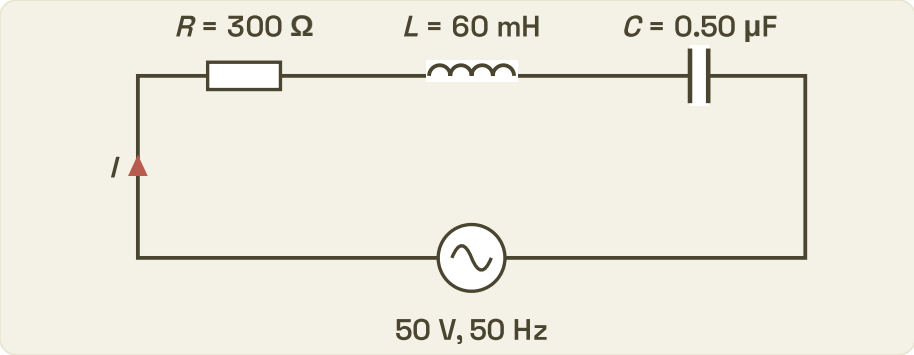


Figure 19.24 *For question 19: a series LCR circuit with $L = 60$ mH, $C = 0.50$ μF and $R = 300\ \Omega$ across a 50 V, 50 Hz a.c. source.*

2. **The heating goes as I², which is never negative.** The power is $P = I^2R$, whichever way the current flows. So average the **square** of the current:

$$\langle P \rangle = \langle I^2 \rangle R$$

3. **Match it with a steady current.** A steady current I_{rms} gives power $I_{\text{rms}}^2 R$. For the same heating,

$$I_{\text{rms}}^2 R = \langle I^2 \rangle R \quad \Rightarrow \quad I_{\text{rms}} = \sqrt{\langle I^2 \rangle}$$

That is the name read backwards: the **root** of the **mean** of the **square**.

4. **Put in the sine:**

$$I_{\text{rms}} = \sqrt{\langle I_0^2 \sin^2 \omega t \rangle} = I_0 \sqrt{\langle \sin^2 \omega t \rangle}$$

5. **The mean of sin² over a cycle is ½.** Since $\sin^2 \omega t = \tfrac{1}{2}(1 - \cos 2\omega t)$, and $\cos 2\omega t$ averages to zero over a cycle, $\langle \sin^2 \omega t \rangle = \tfrac{1}{2}$. (Or: sin² and cos² have the same shape, one shifted from the other, so they have the same average; they add up to 1 at every instant, so each averages ½.)

6. **So**

$$\boxed{I_{\text{rms}} = \frac{I_0}{\sqrt{2}} = 0.707\, I_0} \qquad \text{and similarly} \qquad \boxed{V_{\text{rms}} = \frac{V_0}{\sqrt{2}} = 0.707\, V_0}$$

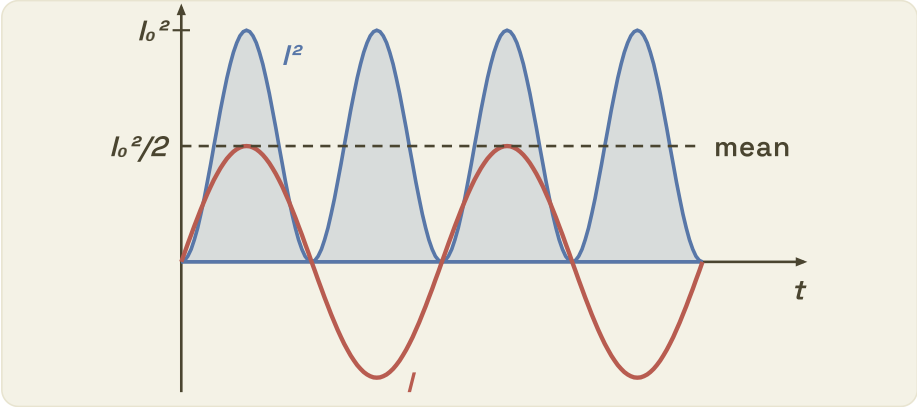


Figure 19.5 *Why the rms value is $I_0/\sqrt{2}$. The current $I = I_0 \sin \omega t$ (red) averages to zero over a cycle. Its square, I^2 (shaded), is never negative: it swings between 0 and I_0^2, and its average, the dashed line, is exactly half way, $I_0^2/2$. The square root of that average is $I_{\text{rms}} = I_0/\sqrt{2}$.*

Mean value over half a cycle

Over a whole cycle the average current is zero, but over **one half cycle** (when the current flows one way) it is not. This **mean value** I_m is what matters for charging a battery through a rectifier, or for electrolysis:

$$I_m = \frac{1}{T/2} \int_0^{T/2} I_0 \sin \omega t \, dt = \frac{2I_0}{T} \left[\frac{-\cos \omega t}{\omega} \right]_0^{T/2} = \frac{2I_0}{\omega T} (1 - \cos \pi) = \frac{2I_0}{2\pi} \times 2$$

since $\omega T = 2\pi$ and $\cos \pi = -1$. So

$$\boxed{I_m = \frac{2I_0}{\pi} = 0.637\, I_0} \qquad \text{and} \qquad V_m = \frac{2V_0}{\pi} = 0.637\, V_0$$

For a sine wave, then, mean (half cycle) < rms < peak: $0.637\, I_0 < 0.707\, I_0 < I_0$. The rms value is larger than the mean because squaring gives the large values of the current more weight.

CAREFUL

Peak, rms and mean are three different numbers

"230 V" for the mains is the **rms** value. The voltage actually swings up to $230\sqrt{2} \approx 325$ V and down to -325 V. Insulation, and a capacitor across the mains, must stand the **peak**, not 230 V. And the mean over a whole cycle is zero; "mean value" in exam questions means the mean over **half** a cycle, $2V_0/\pi$.

WORKED EXAMPLE

Example 19.1: the peak of the mains

The electric mains in a house are marked 220 V. Calculate the peak voltage.

Answer

220 V is the rms value: $V_{\text{rms}} = 220$ V.

$$V_0 = \sqrt{2}\,V_{\text{rms}} = 1.414 \times 220 = 311 \text{ V}$$

So the voltage swings between +311 V and −311 V, and this a.c. heats a resistor exactly as a steady 220 V d.c. supply would. (For 230 V, the peak is 325 V.)

WORKED EXAMPLE

Example 19.2: reading the equation of a current

The current in a circuit is $I = 50 \sin 100\pi t$ amperes. Find (i) the frequency, (ii) the mean value of the current during a positive half cycle, (iii) its virtual (rms) value, and (iv) the current $\frac{1}{300}$ s after it was zero.

Answer

Compare with $I = I_0 \sin \omega t = I_0 \sin 2\pi f t$: $I_0 = 50$ A and $2\pi f = 100\pi$.

i. $f = \dfrac{100\pi}{2\pi} = \mathbf{50\ Hz}$.

ii. Mean over a half cycle: $I_m = \dfrac{2I_0}{\pi} = 0.637 \times 50 = \mathbf{31.8\ A}$.

iii. Virtual (rms) value: $I_v = \dfrac{I_0}{\sqrt{2}} = 0.707 \times 50 = \mathbf{35.4\ A}$.

iv. The current is zero at $t = 0$. At $t = \frac{1}{300}$ s:

$$I = 50 \sin\left(100\pi \times \frac{1}{300}\right) = 50 \sin\frac{\pi}{3} = 50 \times 0.866 = 43.3 \text{ A}$$

WORKED EXAMPLE

Example 19.3: rms value and frequency

An alternating voltage is $V = 100\sqrt{2}\,\sin(100\pi t)$ volts. Find its rms value and its frequency.

Answer

Compare with $V = V_0 \sin \omega t$: $V_0 = 100\sqrt{2}$ V and $\omega = 100\pi$ rad/s.

$$V_{\text{rms}} = \frac{V_0}{\sqrt{2}} = \frac{100\sqrt{2}}{\sqrt{2}} = 100 \text{ V}$$

$$2\pi f = 100\pi \quad \Rightarrow \quad f = 50 \text{ Hz}$$

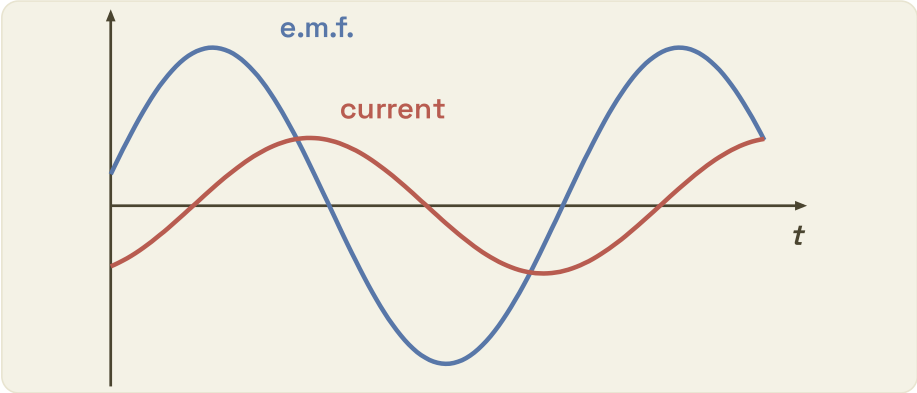


Figure 19.23 For question 17: the e.m.f. (blue, the larger wave) and the current (red) in an LCR circuit against time t.

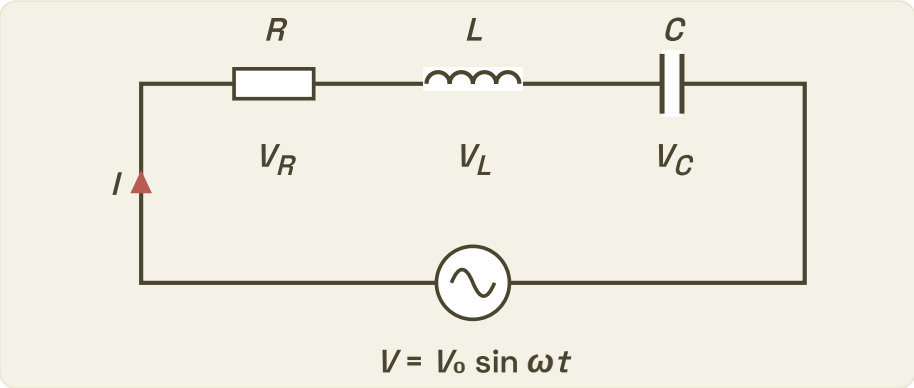


Figure 19.22 *For question 15: a resistor R, an inductor L and a capacitor C in series across an a.c. source $V = V_0 \sin \omega t$, with the p.d.s V_R, V_L, V_C marked across each and V across all three.*

17. The graph shows how the e.m.f. and the current vary with time in an LCR series circuit.
    i. Is the phase constant φ (the angle by which the e.m.f. leads the current) positive or negative? Explain.
    ii. Sketch a phasor diagram for this case.
    iii. Is the circuit more inductive or more capacitive? Explain.
    iv. To increase the rate at which energy is delivered to the resistance, should the inductance be increased or decreased? Justify your answer.

YOUR TURN

The same heating

A steady current of 3 A heats a resistor. What peak value must a sinusoidal a.c. have to heat the same resistor at the same rate?

Answer

The a.c. must have $I_{\text{rms}} = 3$ A, so $I_0 = \sqrt{2} \times 3 = \mathbf{4.24\ A}$.

Check 3

An a.c. ammeter in a circuit reads 2.0 A. The greatest current that actually flows is:

a 1.4 A b 2.0 A
c 2.8 A d 4.0 A

19.4 Phasor diagram

Adding two alternating quantities, such as the voltages across a resistor and a coil in series, means adding two sine waves that rise and fall at different moments. Doing that with trigonometry at every instant is hard work. Engineers use a picture instead.

A **phasor** is an arrow that turns **anticlockwise** about the origin with the angular frequency ω of the a.c.:

- (a) The **length** of the phasor is the **peak value** of the quantity it stands for (V_0 or I_0; or, scaled by $1/\sqrt{2}$, its rms value).
- (b) Its **projection on the vertical axis** at any instant is the quantity's **instantaneous value**. An arrow of length V_0 at angle ωt to the horizontal has vertical projection $V_0 \sin \omega t$: exactly the a.c.

A **phasor diagram** shows the phasors of two or more alternating quantities **of the same frequency**, drawn at one instant. Since they all turn together at the same rate, the angles between them never change: the angle between two phasors is their **phase difference** ϕ. The one ahead in the direction of turning (anticlockwise) **leads**: it reaches its peak first. Phasors add like vectors, and the sum's vertical projection is the sum of the instantaneous values.

You have met the idea before: in simple harmonic motion, the shadow of a point going round a circle moves as $r \sin \omega t$ Grade 12. A phasor is that turning radius.

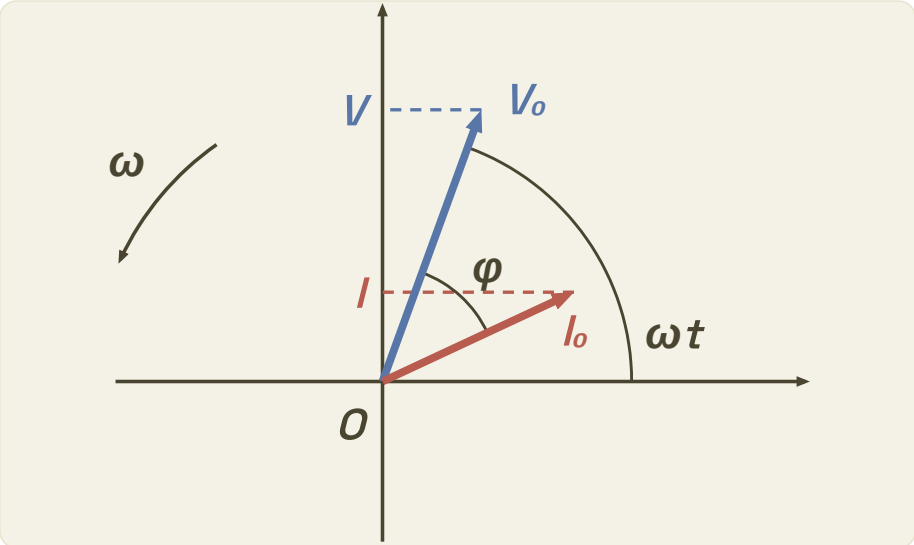


Figure 19.6 *A phasor diagram at one instant. The voltage phasor V_0 makes an angle ωt with the horizontal axis; the current phasor I_0 is an angle ϕ behind it, so the voltage leads the current by ϕ. Both turn anticlockwise at ω; their vertical projections are the instantaneous values.*

Check 4

On a phasor diagram the voltage phasor is drawn 90° anticlockwise from the current phasor. This means the voltage:

a leads the current by 90°, reaching its peak a quarter of a cycle earlier

b lags the current by 90°

c is in phase with the current

d has a peak value 90 times the current's

19.5 A.c. through a resistor

Connect a resistor of resistance R (with no inductance) across an alternating e.m.f. $V = V_0 \sin \omega t$. At every instant the p.d. across the resistor equals the e.m.f., and Ohm's law holds at every instant, so the current is

$$I = \frac{V}{R} = \frac{V_0}{R} \sin \omega t = I_0 \sin \omega t, \qquad I_0 = \frac{V_0}{R}$$

Read the other way, if the current is $I = I_0 \sin \omega t$ (19.1), the p.d. is $V = IR = V_0 \sin \omega t$ (19.2). Either way:

- the current and the e.m.f. are **in phase** ($\phi = 0$): they reach their peaks, pass through zero and reverse **together**;
- the peak (and rms) values obey Ohm's law, $V_0 = I_0 R$, $V_{\text{rms}} = I_{\text{rms}} R$;
- R does not depend on the frequency.

On a phasor diagram the two phasors lie along the **same line**, turning together.

12. a. What is the power factor of a purely capacitive circuit?
    b. Obtain an expression for the phase relationship between the current and the voltage in a purely capacitive circuit.
    c. Why is a purely capacitive circuit an open circuit for d.c.?
    d. An iron-cored coil of 2 H and 50 Ω resistance is placed in series with a resistor of 450 Ω, and a 200 V, 50 Hz a.c. supply is connected across them. Find the current in the coil, its phase angle relative to the supply voltage, and the p.d. across the coil.

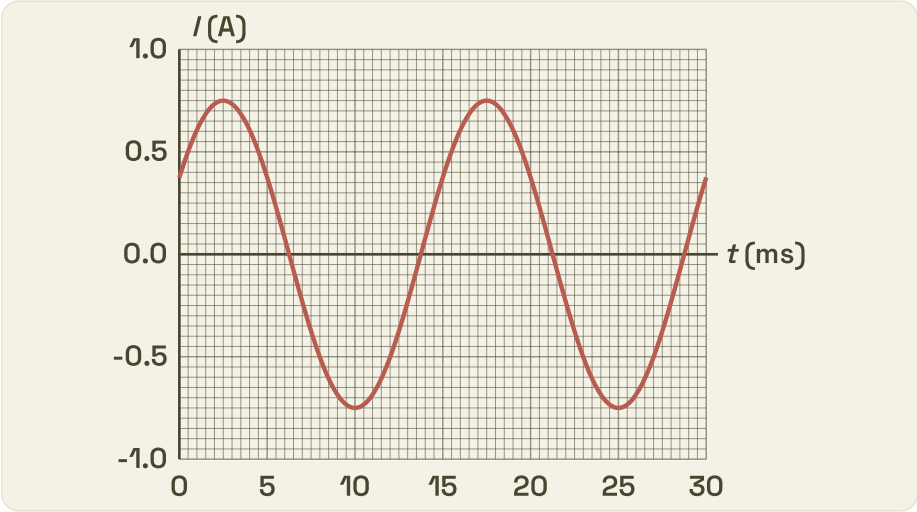


Figure 19.21 *For questions 13 and 18: the current I (in amperes) in a 450 Ω resistor against time t (in milliseconds), from 0 to 30 ms.*

14. a. What is the power factor of a purely inductive circuit?
    b. Obtain an expression for the phase relationship between the current and the voltage in a purely inductive circuit.
    c. Why is a purely inductive circuit a short circuit for d.c.?
    d. A coil with inductance and resistance is connected to an oscillator giving a fixed sinusoidal output of 5 V rms. At 50 Hz the rms current in the coil is 1 A, and at 100 Hz it is 0.625 A. Find the inductance of the coil.

15. An inductor, a capacitor and a resistor are connected in series with a source of alternating voltage, as above.
    a. Write the relationship between the current and the voltage.
    b. Draw a phasor diagram to show the leading or lagging relation between the voltage and the current.
    c. In the series LCR circuit, R = 25 Ω, L = 30 mH and C = 10 μF, connected to a 240 V, 50 Hz a.c. source. Calculate the current and the voltmeter reading across the capacitor.

Check 19

9. The reactance of a capacitor of C μF for a.c. of frequency $\frac{400}{\pi}$ Hz is 25 Ω. The value of C is:

a 100 b 50
c 450 d 25

Check 20

10. What is the phase difference between the rms (virtual) voltage and current when the current in a circuit is wattless?

a 65° b 180°
c 90° d 45°

Answer these questions

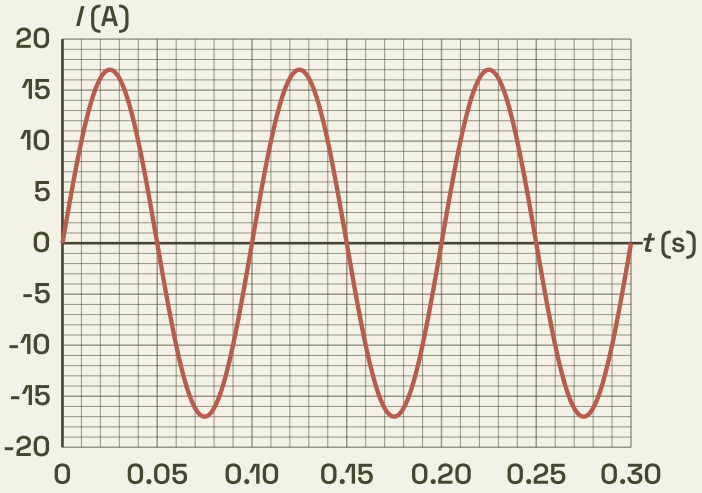


Figure 19.20 *For question 11: an alternating current I (in amperes) against time t (in seconds).*

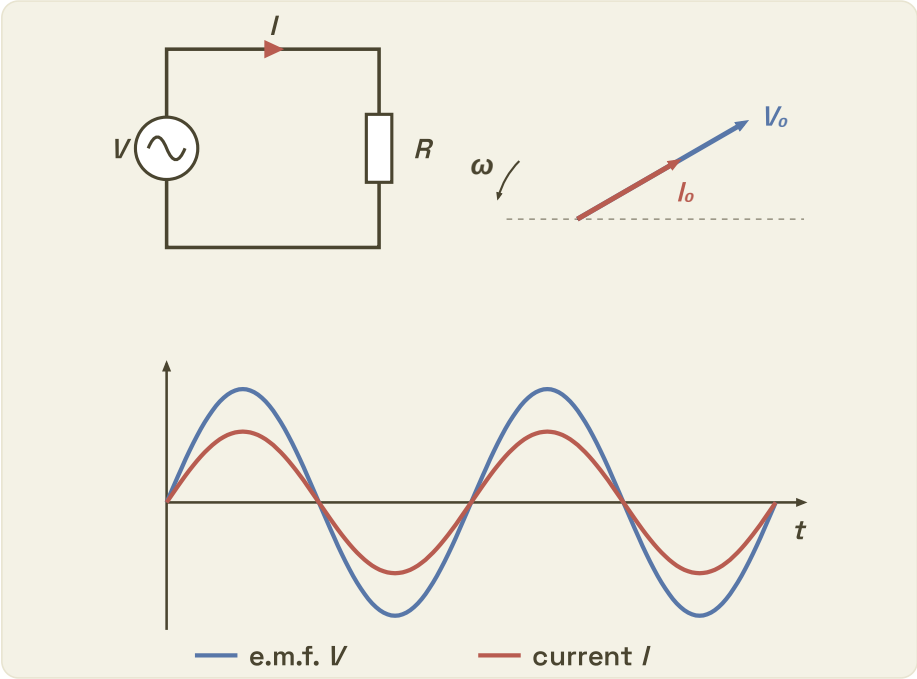


Figure 19.7 *A.c. through a resistor. Top left: the circuit, an a.c. source $V_0 \sin \omega t$ across a resistor R. Top right: the phasors V_0 and I_0 lie along the same line, turning together. Below: the e.m.f. (blue) and the current (red) rise and fall in step: they are in phase.*

19.6 A.c. through an inductor

Now connect a **pure inductor**, a coil of inductance L with no resistance, across the a.c. source. Let the current be

$$I = I_0 \sin \omega t \qquad (19.3)$$

1. **The coil fights every change of current.** A changing current in a coil induces a back e.m.f. $-L\frac{dI}{dt}$ in it (self-induction, Lenz's law: Unit 18).
2. **Kirchhoff's loop law.** With no resistance in the circuit, the source's e.m.f. and the back e.m.f. must add to zero ($IR = 0$):

$$V + \left(-L\frac{dI}{dt}\right) = 0 \quad \Rightarrow \quad V = L\frac{dI}{dt}$$

So the source's voltage is largest when the current is **changing** fastest, not when it is largest.

3. **Differentiate the current:**

$$V = L\frac{d}{dt}(I_0 \sin \omega t) = \omega L\, I_0 \cos \omega t$$

4. **Name the quantity ωL.** It has the dimensions of resistance (volts per ampere: ωL in $\mathrm{s}^{-1} \times \mathrm{V\ s/A} = \mathrm{V/A} = \Omega$). It is the **inductive reactance**, the inductor's opposition to a.c.:

$$\boxed{X_L = \omega L = 2\pi f L}$$

5. **So** $V = I_0 X_L \cos\omega t$. Writing $V_0 = I_0 X_L$ and using $\cos\omega t = \sin\left(\omega t + \frac{\pi}{2}\right)$:

$$V = V_0 \cos\omega t = V_0 \sin\left(\omega t + \frac{\pi}{2}\right) \qquad (19.4)$$

Comparing (19.3) and (19.4): **in a pure inductor the e.m.f. leads the current by 90° (π/2)**. Said the other way, the current **lags** the voltage by a quarter of a cycle. The voltage peaks when the current is passing through zero (changing fastest), and the current peaks a quarter of a cycle later, when the voltage has fallen to zero.

The peak values obey $I_0 = \frac{V_0}{X_L}$, and the rms values $I_{\text{rms}} = \frac{V_{\text{rms}}}{X_L}$: "Ohm's law" with the reactance in place of R, for peak or rms values (never for instantaneous ones, which are out of step).

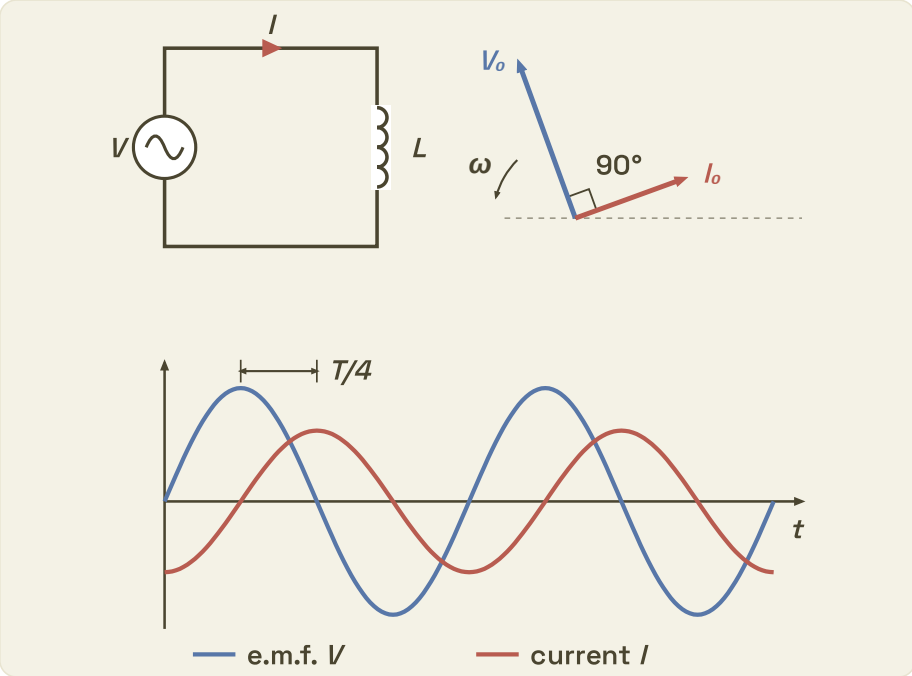


Figure 19.8 *A.c. through a pure inductor. The circuit; the phasors, with V_0 90° ahead of I_0 (anticlockwise); and the waves: the e.m.f. (blue) reaches each peak a quarter of a cycle before the current (red). The e.m.f. leads; the current lags.*

Reactance and frequency

$X_L = 2\pi f L$ grows in **proportion to the frequency**. So:

- for d.c. (f = 0), $X_L = 0$: an ideal coil is just a piece of wire, a **short circuit for d.c.** (a real coil has only the small resistance of its wire);
- as the frequency rises, X_L rises and the current $I_{\text{rms}} = \frac{V_{\text{rms}}}{2\pi f L}$ falls: a coil **lets low frequencies through and holds back high ones**;
- at very high frequency, $X_L \to \infty$ and the coil passes almost nothing.

That is why a coil is called a **choke** in a.c. work: it chokes a.c. while letting d.c. through.

Check 12

2. In a series LCR circuit the resistance is 2 Ω, the capacitive reactance 4 Ω and the inductive reactance 4 Ω, on a 12 V supply. The current is:

a 2 A
b 6 A
c 12 A
d 1.2 A

Check 13

3. In an inductor (a coil made to be an inductor), the reactance X_L and the resistance R are related by:

a $X_L = R$
b $X_L \gg R$
c $X_L \ll R$
d $X_L = \infty$

Check 14

4. The resonant frequency of a circuit is f. If the capacitance is made 4 times as large, the resonant frequency becomes:

a f/2
b 2f
c f
d f/4

Check 15

5. The ratio of the mean value of a.c. over half a cycle to its rms value is:

a 2 : π
b 2√2 : π
c √2 : π
d √2 : 1

Check 16

6. In an LCR circuit the p.d. across the inductance is 60 V, across the capacitor 30 V and across the resistance 40 V. The supply voltage is:

a 50 V
b 70 V
c 130 V
d 10 V

Check 17

7. A capacitor of capacitance 1 μF is charged to a potential of 1 V and then connected across an inductor of inductance 1 μH. The maximum current that flows in the circuit is:

a √100 mA
b 1 mA
c 1 μA
d 1000 mA

Check 18

8. A step-down transformer is connected to a 2400 V line, and a current of 80 A flows in its output load. The ratio of turns in the primary and secondary coils is 20 : 1. If the transformer is 100% efficient, the current in the primary coil is:

a 1600 A
b 20 A
c 4 A
d 1.5 A

WORKED EXAMPLE

Example 19.19: wattless current

What do you mean by wattless current?

Answer

A current is **wattless** if it uses no power in the circuit, although it flows. This happens when the phase difference between the current and the e.m.f. is 90° (π/2). Since $P_{\text{av}} = V_{\text{rms}} I_{\text{rms}} \cos\phi$, with $\phi = \pi/2$:

$$P_{\text{av}} = V_{\text{rms}} I_{\text{rms}} \cos\frac{\pi}{2} = 0$$

So the current in a pure inductor (or a pure capacitor) is wattless: no power is used in it.

Check 10

A coil draws 2 A from a 230 V a.c. supply, and a wattmeter shows it uses 92 W. Its power factor is:

a 0.92 b 0.2
c 0.4 d 1

IN SHORT

- **A.c.:** $V = V_0 \sin\omega t$, $I = I_0 \sin\omega t$, $\omega = 2\pi f$, $T = 1/f$. Nepal's mains: 50 Hz, about 230 V rms.
- **rms (virtual) value:** the steady value that heats a resistance equally. $I_{\text{rms}} = I_0/\sqrt{2} = 0.707\, I_0$. **Mean over half a cycle:** $2I_0/\pi = 0.637\, I_0$.
- **Phasor:** an arrow of length = peak value turning anticlockwise at ω; its vertical projection is the instantaneous value; the angle between phasors is the phase difference.
- **R:** V and I in phase. **L:** $X_L = 2\pi fL$, V leads I by 90°, short circuit for d.c. **C:** $X_C = 1/(2\pi fC)$, I leads V by 90°, open circuit for d.c.
- **Series LCR:** $Z = \sqrt{R^2 + (X_L - X_C)^2}$, $\tan\phi = (X_L - X_C)/R$ (φ = the angle by which V leads I). LR and CR are the cases $X_C = 0$, $X_L = 0$.
- **Resonance:** $X_L = X_C$, $f_0 = \dfrac{1}{2\pi\sqrt{LC}}$; Z = R (least), current greatest, φ = 0.
- **Quality factor:** $Q = \dfrac{V_L}{V} = \dfrac{\omega_0 L}{R} = \dfrac{1}{R}\sqrt{\dfrac{L}{C}}$; sharpness of resonance, $\Delta f = f_0/Q$.
- **Power:** $P_{\text{av}} = V_{\text{rms}} I_{\text{rms}} \cos\phi$; power factor $\cos\phi = R/Z$. Pure L or C: $\phi = 90°$, no power used: **wattless current**.

Exercise

Choose the correct option

Check 11

1. When 100 V d.c. is applied across a coil, a current of 1 A flows through it. When 100 V a.c. at 50 Hz is applied to the same coil, only 0.5 A flows. The inductance of the coil is:

a 0.55 mH b 5.5 mH
c 0.55 H d 55 mH

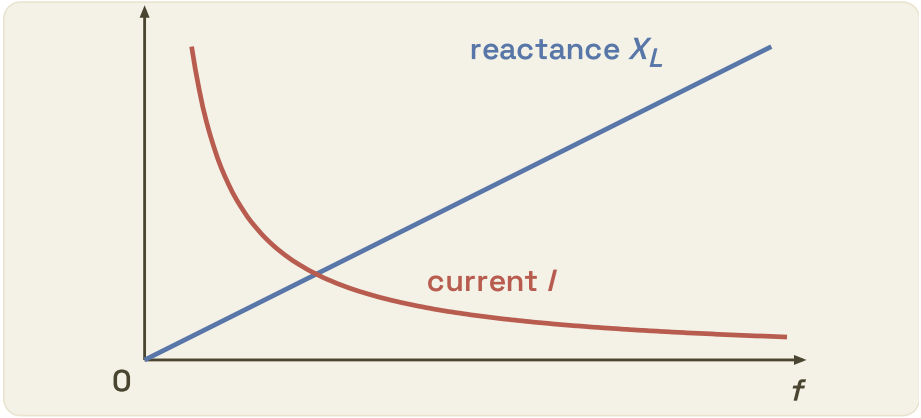


Figure 19.9 *For a coil of fixed L across a fixed voltage: the inductive reactance $X_L = 2\pi fL$ (blue) is a straight line through the origin, and the current V/X_L (red) falls as 1/f.*

Notice the shapes in Figure 19.9: the reactance is a straight line through the origin (it is proportional to f), and the current is a hyperbola (it is proportional to 1/f), very large at low frequency and falling towards zero at high frequency.

WORKED EXAMPLE

Example 19.4: a coil on the mains

An inductor of negligible resistance has a reactance of 22 Ω at 200 Hz. It is connected to a 220 V, 50 Hz power line. Find its inductance and the current.

Answer

At 200 Hz, $X_L = 2\pi fL = 22\ \Omega$, so

$$L = \frac{22}{2\pi \times 200} = \frac{22}{2 \times \frac{22}{7} \times 200} = \frac{7}{400} = 0.0175\ \text{H}$$

At 50 Hz the reactance is a quarter as much (it is proportional to f):

$$X_L = 2\pi \times 50 \times 0.0175 = 5.5\ \Omega$$

$$I = \frac{V}{X_L} = \frac{220}{5.5} = 40\ \text{A (rms)}$$

Check 5

The frequency of the supply across a pure inductor is doubled, the voltage kept the same. The current:

a doubles b halves
c stays the same d becomes four times as large

19.7 A.c. through a capacitor

Connect a capacitor of capacitance C across the a.c. source. A capacitor's plates are separated by an insulator, so no charge ever crosses the gap. But as the voltage rises and falls, charge flows **onto** one plate and **off** the other, back and forth through the wires, and that back-and-forth flow is a real alternating current in the circuit.

Let the current in the wires be

$$I = I_0 \sin \omega t \qquad (19.5)$$

1. **The capacitor's p.d.** With charge q on its plates, $V_C = \dfrac{q}{C}$ (19.6).
2. **Charge from current.** The current is the rate at which charge arrives: $I = \dfrac{dq}{dt}$, so $dq = I\,dt = I_0 \sin \omega t\,dt$.
3. **Integrate** (taking q to swing evenly about zero, so no constant is left over):
$$q = \int I_0 \sin \omega t\,dt = -\frac{I_0}{\omega} \cos \omega t$$
4. **So the p.d. is**
$$V_C = \frac{q}{C} = -\frac{I_0}{\omega C} \cos \omega t$$
5. **Name 1/(ωC).** It too has the dimensions of resistance. It is the **capacitive reactance**:
$$\boxed{X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}}$$
6. The source's e.m.f. equals the capacitor's p.d. at every instant, $V = V_C = -I_0 X_C \cos \omega t$. With $V_0 = I_0 X_C$ and $-\cos \omega t = \sin\left(\omega t - \frac{\pi}{2}\right)$:
$$V = -V_0 \cos \omega t = V_0 \sin\left(\omega t - \frac{\pi}{2}\right) \qquad (19.7)$$

Comparing (19.5) and (19.7): **in a capacitor the e.m.f. lags behind the current by 90°**, or the **current leads** the voltage by a quarter of a cycle. It makes sense: the current is largest when the capacitor is empty and charge rushes on most easily; by the time the voltage has reached its peak, the capacitor is full and the current has stopped.

Again $I_0 = \dfrac{V_0}{X_C}$ and $I_{\text{rms}} = \dfrac{V_{\text{rms}}}{X_C}$.

Reactance and frequency

$X_C = \dfrac{1}{2\pi f C}$ is **inversely proportional to the frequency**:

- for d.c. (f = 0), $X_C \to \infty$: a capacitor is an **open circuit for d.c.** Once it has charged, no steady current flows, since no charge crosses the insulator;
- as the frequency rises, X_C falls and the current rises: a capacitor **lets high frequencies through and blocks low ones**, the opposite of a coil.

2. **Use the identity** $\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$:
$$P = \tfrac{1}{2} V_0 I_0 \left[\cos \phi - \cos(2\omega t - \phi)\right]$$
The first term is steady; the second swings at twice the supply frequency.
3. **Average over a cycle.** The second term averages to zero, leaving
$$\langle P \rangle = \tfrac{1}{2} V_0 I_0 \cos \phi = \frac{V_0}{\sqrt{2}} \cdot \frac{I_0}{\sqrt{2}} \cos \phi$$
$$\boxed{P_{\text{av}} = V_{\text{rms}}\, I_{\text{rms}} \cos \phi} \qquad (19.21)$$

The peak power, when V and I are in phase, is $P_0 = V_0 I_0$, and the power then swings between 0 and P_0, averaging $P_0/2$. When they are out of phase, P is **negative** for part of each cycle: energy flows **back** from the circuit to the source, and the average is smaller.

- $V_{\text{rms}} I_{\text{rms}}$, what a voltmeter and an ammeter seem to promise, is the **apparent power**, in volt-amperes (VA);
- $\cos \phi$ is the **power factor**;
- so **average (real) power = apparent power × power factor**, in watts.

Power factor

The **power factor** is the ratio of the real (useful) power to the apparent power:

$$\text{power factor} = \cos \phi = \frac{\text{real power}}{\text{apparent power}} = \frac{I_{\text{rms}}^2 R}{I_{\text{rms}}^2 Z} = \frac{R}{Z} = \frac{R}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}$$

It is the cosine of the phase angle between the voltage and the current. The phase angle φ lies between −90° (−π/2, all capacitance) and +90° (+π/2, all inductance), so cos φ is never negative: the power factor lies between 0 and 1. A power factor of 1 (100%) means every volt-ampere supplied does useful work; a low power factor means a large current flows back and forth for little real power.

Why does that matter, if the energy that flows back isn't used? Because the current still flows in the wires. A factory whose motors (coils) give it a power factor of 0.5 draws **twice** the current it would at a power factor of 1 for the same real power, and the supply lines and transformers lose I^2R, four times as much heat. That is why electricity suppliers want large users to keep their power factor high, and factories put banks of capacitors beside their motors: the capacitors' leading current cancels part of the motors' lagging current, bringing φ closer to zero.

Special cases

1. **Pure resistance**, $\phi = 0$: $P_{\text{av}} = V_{\text{rms}} I_{\text{rms}} \cos 0^\circ = V_{\text{rms}} I_{\text{rms}}$. The power factor is 1 and all the power supplied is used, as heat.
2. **Pure capacitance**, $\phi = -90^\circ$: $P_{\text{av}} = V_{\text{rms}} I_{\text{rms}} \cos 90^\circ = 0$. The capacitor takes energy into its electric field for a quarter cycle and gives it all back in the next.
3. **Pure inductance**, $\phi = 90^\circ$: $P_{\text{av}} = 0$. The coil stores energy in its magnetic field and returns it.

So the current through a pure inductor or a pure capacitor uses **no power** on average. Such a current is called a **wattless current**. That is the idea behind a **choke coil**: an inductor (with very little resistance) in series with a fluorescent tube or a fan's speed regulator limits the current, as a resistor would, but wastes almost no energy as heat.

WORKED EXAMPLE

Example 19.17: three voltmeter readings

An inductor, a resistor and a capacitor are connected in series across an a.c. supply. A voltmeter reads 60 V across the inductor, 16 V across the resistor and 30 V across the capacitor. (i) What will it read across the whole series circuit? (ii) What is the power factor?

Answer

(i) The p.d.s add as phasors:

$$V = \sqrt{V_R^2 + (V_L - V_C)^2} = \sqrt{16^2 + (60-30)^2} = \sqrt{256 + 900} = \sqrt{1156} = 34 \text{ V}$$

(Not $60 + 16 + 30 = 106$ V.)

(ii)

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{30}{16} = 1.875, \qquad \phi = 61.9^\circ$$

$$\cos\phi = \frac{V_R}{V} = \frac{16}{34} = 0.47$$

WORKED EXAMPLE

Example 19.18: tuning to 1 MHz

A circuit has an inductor of 200 μH and a resistance of 10 Ω in series with a variable capacitor, across a 0.10 V (rms), 1.0 MHz supply. Calculate (i) the capacitance for resonance, (ii) the quality factor at resonance.

Answer

(i) At resonance $f = \dfrac{1}{2\pi\sqrt{LC}}$, so

$$C = \frac{1}{4\pi^2 f^2 L} = \frac{1}{4\pi^2 \times (10^6)^2 \times 2 \times 10^{-4}} = 1.27 \times 10^{-10} \text{ F} = 127 \text{ pF}$$

(ii)

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{10}\sqrt{\frac{2 \times 10^{-4}}{1.27 \times 10^{-10}}} = \frac{1257}{10} = 126$$

Check: $Q = \dfrac{2\pi f L}{R} = \dfrac{2\pi \times 10^6 \times 2 \times 10^{-4}}{10} = 126$. So at resonance the p.d. across the capacitor is about $126 \times 0.10 = 12.6$ V, from a 0.10 V supply.

19.13 Power in an a.c. circuit

In an a.c. circuit both the e.m.f. and the current change all the time, so the power, the rate at which energy is delivered, changes too. The **instantaneous power** is the product of the instantaneous e.m.f. and the instantaneous current:

$$P = VI$$

What matters for a bill or for heating is its **average** over a cycle.

The average power

Let the e.m.f. be $V = V_0 \sin\omega t$ and the current $I = I_0 \sin(\omega t - \phi)$, lagging by φ (for a leading current φ is negative; the result is the same).

1. **Instantaneous power:**

$$P = V_0 I_0 \sin\omega t \, \sin(\omega t - \phi)$$

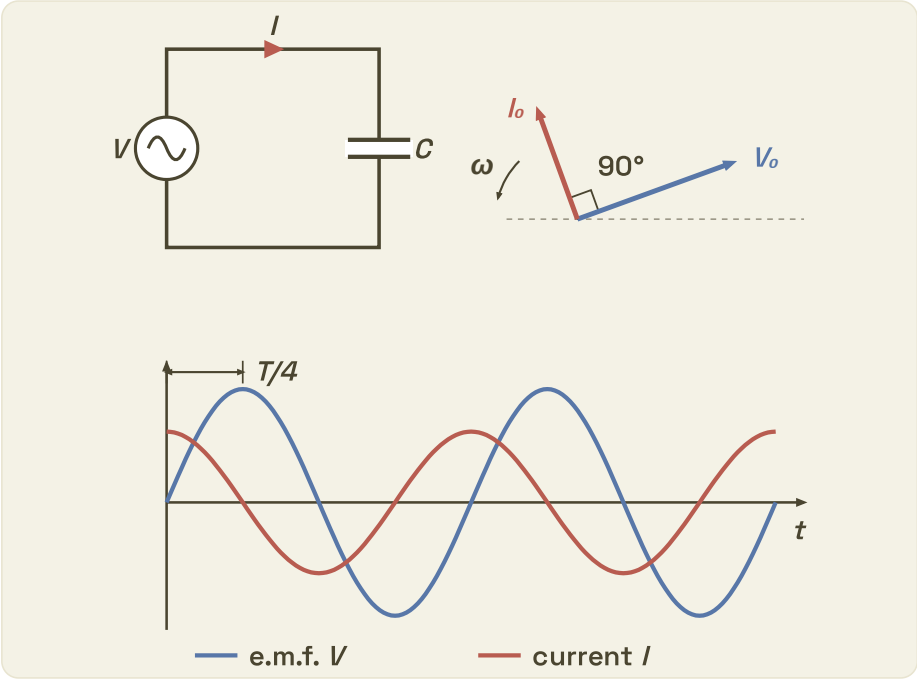


Figure 19.10 A.c. through a capacitor. The circuit; the phasors, with I_0 90° ahead of V_0; and the waves: the current (red) reaches each peak a quarter of a cycle before the e.m.f. (blue). The current leads; the e.m.f. lags.

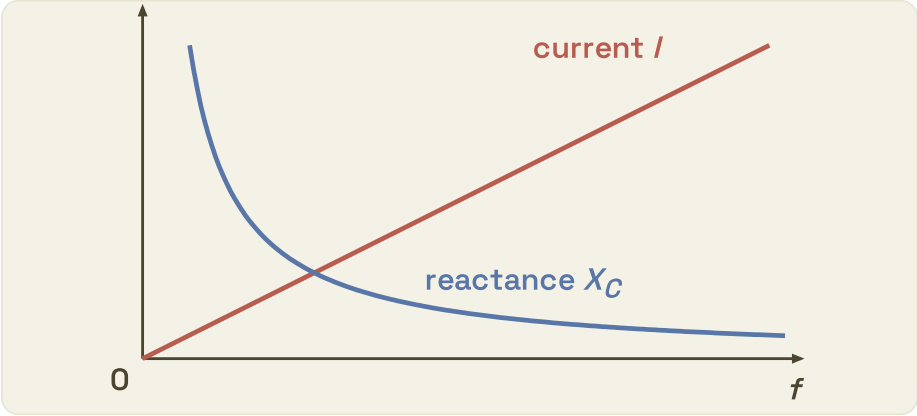


Figure 19.11 For a capacitor of fixed C across a fixed voltage: the capacitive reactance $X_C = 1/(2\pi fC)$ (blue) falls as 1/f, and the current $V/X_C = 2\pi fCV$ (red) rises in proportion to f.

So a capacitor and a coil are opposites in every way. A coil passes d.c. and blocks high frequencies; a capacitor blocks d.c. and passes high frequencies. A loudspeaker box uses exactly this to send the bass to the big speaker (through a coil) and the treble to the small one (through a capacitor).

	Resistor	Pure inductor	Capacitor
Opposition	resistance R	$X_L = \omega L = 2\pi f L$	$X_C = \dfrac{1}{\omega C} = \dfrac{1}{2\pi f C}$
Change with frequency	none	$\propto$ f	$\propto$ 1/f
For d.c. (f = 0)	R	zero: a short circuit	infinite: an open circuit
Phase	V and I in phase	V leads I by 90°	I leads V by 90°
Peak current	V_0/R	V_0/X_L	V_0/X_C
Average power used	$V_{\text{rms}} I_{\text{rms}}$	zero	zero

CAREFUL

CIVIL: who leads?

A memory aid: **CIVIL**. In a **C**apacitor, **I** comes before **V**; **V** comes before **I** in an inductor (**L**).

Check 6

Which of these lets a steady d.c. pass, but holds back a high-frequency a.c.?

a a capacitor
b an inductor
c neither
d both

19.8 A.c. through an inductance and a resistance (LR circuit)

A real coil has both inductance and the resistance of its wire, so the LR circuit is the one you meet most often. Connect a resistance R and an inductance L **in series** across an a.c. source of e.m.f. V and frequency f, and let the current be $I = I_0 \sin \omega t$.

In series, the **same current** flows through both, so draw the current's phasor first, along the horizontal (OA). Then:

- the p.d. across R, $V_R = IR$, is **in phase** with the current: along OA;
- the p.d. across L, $V_L = IX_L$, **leads** the current by 90°: straight up (AB).

The supply's e.m.f. is the sum of the two p.d.s at every instant, so its phasor is the **vector sum** OB. Since the two are at right angles, in the right-angled triangle OAB:

$$OB^2 = OA^2 + AB^2 \quad \Rightarrow \quad V^2 = V_R^2 + V_L^2 = (IR)^2 + (IX_L)^2 = I^2\left(R^2 + X_L^2\right)$$

$$I = \frac{V}{\sqrt{R^2 + X_L^2}} \qquad (19.9)$$

The total opposition of the circuit to a.c., $\dfrac{V}{I}$, is its **impedance** Z:

$$\boxed{Z = \sqrt{R^2 + X_L^2} = \sqrt{R^2 + (2\pi f L)^2}} \qquad (19.10)$$

measured in ohms. The **phase angle** ϕ by which the e.m.f. leads the current is the angle AOB:

$$\tan\phi = \frac{AB}{OA} = \frac{V_L}{V_R} = \frac{IX_L}{IR} = \frac{X_L}{R}, \qquad \boxed{\phi = \tan^{-1}\frac{X_L}{R}} \qquad (19.11)$$

So in an LR circuit the e.m.f. leads the current by an angle between 0 (all resistance) and 90° (all inductance).

WORKED EXAMPLE

Example 19.14: an iron-cored coil with a resistor

An iron-cored coil of inductance 2 H and resistance 50 Ω is placed in series with a resistor of 450 Ω, and a 200 V, 50 Hz a.c. supply is connected across them. Find (i) the current in the coil, (ii) its phase angle relative to the supply voltage, (iii) the p.d. across the coil.

Answer

The total resistance is $R + r = 450 + 50 = 500\ \Omega$, and $X_L = 2\pi f L = 2\pi \times 50 \times 2 = 628.3\ \Omega$.

(i)

$$Z = \sqrt{(R+r)^2 + X_L^2} = \sqrt{500^2 + 628.3^2} = \sqrt{644\,784} = 803\ \Omega$$

$$I = \frac{V}{Z} = \frac{200}{803} = 0.249\ \text{A} \approx 0.25\ \text{A}$$

(ii) The phase angle uses the **whole** resistance of the circuit:

$$\tan\phi = \frac{X_L}{R+r} = \frac{628.3}{500} = 1.257, \qquad \phi = 51.5^\circ$$

The current lags the supply voltage by 51.5°.

(iii) The coil has both resistance and reactance, so its own impedance is $Z_{\text{coil}} = \sqrt{r^2 + X_L^2} = \sqrt{50^2 + 628.3^2} = 630.3\ \Omega$, and

$$V_{\text{coil}} = IZ_{\text{coil}} = 0.249 \times 630.3 = 157\ \text{V}$$

(Here r is small beside X_L, so $IX_L = 0.249 \times 628.3 = 156$ V comes out almost the same.)

WORKED EXAMPLE

Example 19.15: a large phase shift

An LCR series circuit with L = 1 H, C = 1 μF and R = 100 Ω is connected to a 50 Hz source. What is the phase difference between the current and the voltage?

Answer

$$X_L = 2\pi f L = 100\pi = 314.2\ \Omega, \qquad X_C = \frac{1}{2\pi f C} = \frac{10^6}{100\pi} = 3183.1\ \Omega$$

$$\tan\phi = \frac{X_L - X_C}{R} = \frac{314.2 - 3183.1}{100} = \frac{-2868.9}{100} = -28.69$$

$$\phi = -88.0^\circ$$

The phase difference is 88°, with the **voltage lagging** (the current leading): at 50 Hz this circuit is far below its resonant frequency (159 Hz), so it is strongly capacitive.

WORKED EXAMPLE

Example 19.16: another iron-cored coil

An iron-cored coil of inductance 3 H and resistance 50 Ω is in series with a resistor of 550 Ω, across a 100 V, 50 Hz supply. Find the current in the coil and the p.d. across the coil.

Answer

$R + r = 550 + 50 = 600\ \Omega$, $X_L = 2\pi \times 50 \times 3 = 942.5\ \Omega$.

$$Z = \sqrt{600^2 + 942.5^2} = \sqrt{1\,248\,264} = 1117\ \Omega, \qquad I = \frac{100}{1117} = 0.0895\ \text{A} \approx 0.09\ \text{A}$$

The coil's impedance is $\sqrt{50^2 + 942.5^2} = 943.8\ \Omega$, so

$$V_{\text{coil}} = 0.0895 \times 943.8 = 84.5\ \text{V} \approx 85\ \text{V}$$

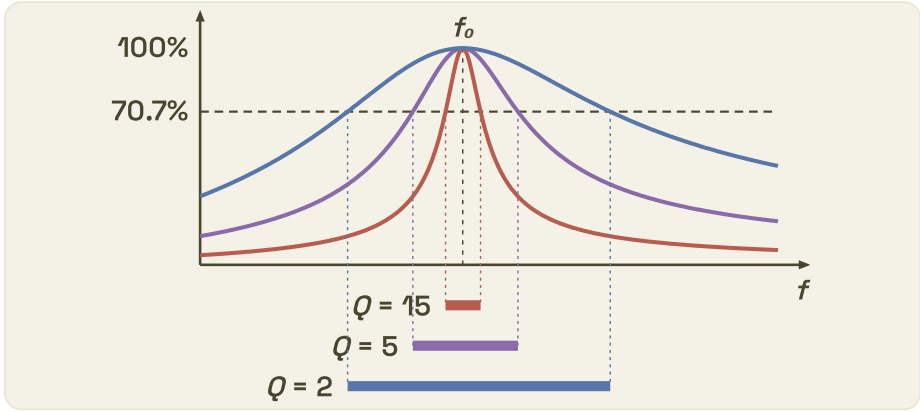


Figure 19.19 *The current near resonance for three values of Q, each drawn as a share of its own peak, so their widths can be compared. The dashed line is 1/√2 (70.7%) of the peak; the bars below the axis show each curve's width there, its bandwidth $\Delta f = f_0/Q$. High Q: narrow; low Q: broad.*

Some important facts

1. A coil with zero resistance uses no power (Section 19.13).
2. A circuit containing a capacitor only uses no power.
3. A series LCR circuit is used as a radio's **tuner**.
4. The quantities $\frac{L}{R}$, CR and $\sqrt{LC}$ all have the **dimensions of time** (they are measured in seconds): L/R is the time a coil's current takes to build up, CR the time a capacitor takes to charge, and $2\pi\sqrt{LC}$ the period of an LC circuit's natural oscillation.
5. Self-induction is what makes a **choke coil** work (below).
6. A.c. can be changed to d.c. by a **rectifier**, and d.c. to a.c. by an **inverter** (an oscillator with a switching circuit).

CAREFUL

Two common wrong statements

- Some books say "a highly inductive circuit has a low power factor and a highly capacitive circuit has a high power factor." Not so: power factor is $\cos\phi$, and a circuit that is mostly reactance of **either** kind has φ near ±90°, so its power factor is near zero, **low**. Only a mostly resistive circuit (or one at resonance) has a power factor near 1.
- Some books say that in (analog) television the sound was sent by amplitude modulation and the picture by frequency modulation. It was the other way round: the picture was amplitude-modulated and the sound frequency-modulated.

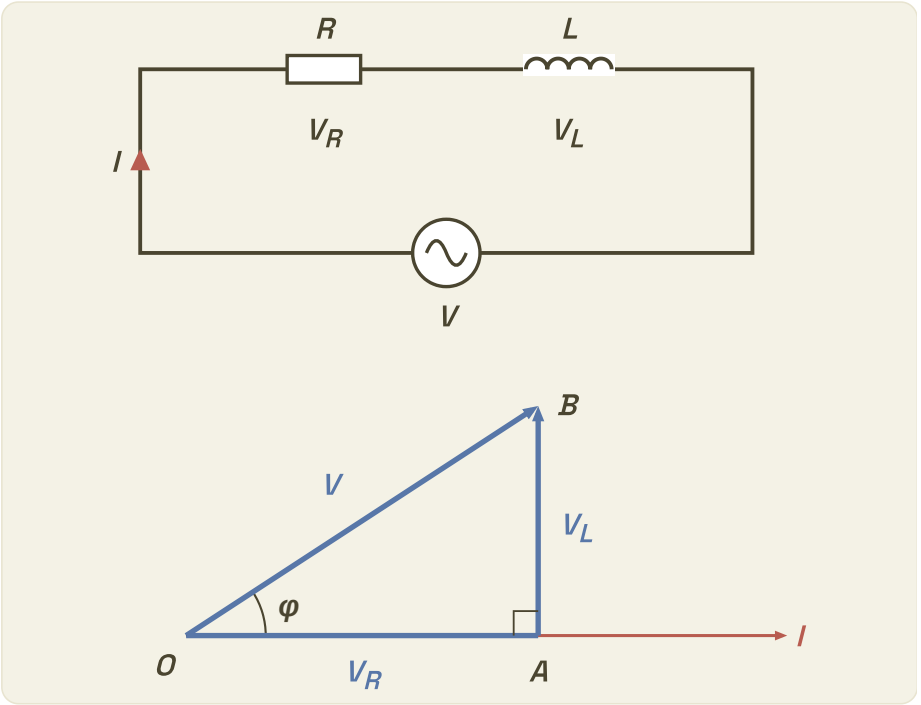


Figure 19.12 *An LR series circuit and its phasor diagram. The current I is the reference, along OA. V_R (OA) is in phase with I; V_L (AB) leads it by 90°; the supply's e.m.f. V (OB) is their vector sum, ahead of the current by the angle φ, with $\tan\phi = X_L/R$.*

CAREFUL

The p.d.s don't add up as numbers

In an LR circuit on a 10 V supply, a voltmeter may read 6 V across R and 8 V across L. Six and eight make fourteen, not ten! There is no mistake: the two p.d.s peak at different moments, so they add as **phasors**, $\sqrt{6^2+8^2} = 10$ V. In the same way $Z \neq R + X_L$; it is $\sqrt{R^2+X_L^2}$.

WORKED EXAMPLE

Example 19.5: resistance, impedance and inductance of a coil

When 100 V d.c. is applied across a coil, a current of 1 A flows through it. When 100 V a.c. at 50 Hz is applied to the same coil, only 0.5 A flows. Calculate the resistance, the impedance and the inductance of the coil.

Answer

With **d.c.** the coil's inductance plays no part ($X_L = 0$): $R = \dfrac{V}{I} = \dfrac{100}{1} = 100\ \Omega$.

With **a.c.**: $Z = \dfrac{V}{I} = \dfrac{100}{0.5} = 200\ \Omega$.

For an LR circuit, $Z^2 = R^2 + X_L^2$:

$$X_L^2 = 200^2 - 100^2 = 40\,000 - 10\,000 = 30\,000, \qquad X_L = 100\sqrt{3} = 173.2\ \Omega$$

$$L = \frac{X_L}{2\pi f} = \frac{173.2}{2 \times 3.14 \times 50} = 0.55\ \text{H}$$

WORKED EXAMPLE

Example 19.6: the same supply, three elements

An alternating voltage of 10 V rms and frequency 50 Hz is applied in turn to (i) a resistor of 5 Ω, (ii) a pure inductor of 2 H, and (iii) a capacitor of 1 μF. Find the rms current in each case.

Answer

i. $I_{\text{rms}} = \dfrac{V_{\text{rms}}}{R} = \dfrac{10}{5} = \mathbf{2\ A}$.

ii. $X_L = 2\pi f L = 2\pi \times 50 \times 2 = 628\ \Omega$, so $I_{\text{rms}} = \dfrac{10}{628} = \mathbf{0.016\ A}$ (16 mA).

iii. $X_C = \dfrac{1}{2\pi f C} = \dfrac{1}{2\pi \times 50 \times 10^{-6}} = 3183\ \Omega$, so $I_{\text{rms}} = \dfrac{V_{\text{rms}}}{X_C} = V_{\text{rms}} \times 2\pi f C = 10 \times 2\pi \times 50 \times 10^{-6} = \mathbf{0.0031\ A}$ (3.1 mA).

WORKED EXAMPLE

Example 19.7: power in an LR circuit

A.c. mains of 200 V, 50 Hz are connected to an inductance of 100 mH and a resistance of 20 Ω in series. Calculate the power consumed.

Answer

$L = 0.1$ H, so $X_L = 2\pi \times 50 \times 0.1 = 31.4\ \Omega$.

$$Z = \sqrt{R^2 + X_L^2} = \sqrt{20^2 + 31.4^2} = \sqrt{400 + 987} = 37.24\ \Omega$$

$$I = \frac{V}{Z} = \frac{200}{37.24} = 5.37\ \text{A}$$

Only the resistance uses power (an ideal coil uses none on average, Section 19.13):

$$P = I^2 R = 5.37^2 \times 20 = 577\ \text{W}$$

DID YOU KNOW?

Why the radio dial works

The aerial of a medium-wave radio picks up every station broadcasting in range, from about 530 kHz to 1600 kHz, all at once, as tiny currents of different frequencies. The tuning circuit's resonance makes one of them hundreds of times larger than the rest. In the same way, a mobile phone, a Wi-Fi router, a TV, and the metal detector at the airport all depend on LC circuits tuned to one frequency.

19.12 Quality factor of an LCR series circuit

At resonance something surprising happens. The current is large, $I = V/R$, and with a small R it can be very large; so the p.d. across the coil, $V_L = IX_L$, and across the capacitor, $V_C = IX_C$, can be **many times larger than the applied e.m.f.** They cancel each other, so the supply never sees them, but a voltmeter across the capacitor alone would. A circuit with a small R on a 10 V supply can have hundreds of volts across its capacitor.

The **quality factor** (Q-factor) of a series LCR circuit measures this voltage magnification. It is the ratio of the p.d. across L (or across C) at resonance to the applied e.m.f.:

$$Q = \frac{\text{p.d. across L or C}}{\text{applied e.m.f.}} = \frac{V_L}{V} = \frac{I_r X_L}{I_r R} = \frac{X_L}{R} \quad \left(\text{or } \frac{X_C}{R}\right)$$

where $I_r = V/R$ is the current at resonance. At resonance $\omega = \dfrac{1}{\sqrt{LC}}$, so

$$Q = \frac{\omega L}{R} = \frac{L}{R\sqrt{LC}} \qquad \Rightarrow \qquad \boxed{Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R} = \frac{1}{R}\sqrt{\frac{L}{C}}}$$

The quality factor also tells how **sharp** the resonance curve is: a high Q means a tall, narrow peak, which picks out one frequency and rejects its neighbors; a low Q, a low, broad one. One exact way to say it: the **bandwidth** Δf, the width of the peak between the two frequencies where the power has fallen to half its peak value (the current to $1/\sqrt{2}$ of its peak), is

$$\Delta f = \frac{f_0}{Q}$$

So a radio's tuning circuit wants a large Q (a small resistance) to separate stations close together.

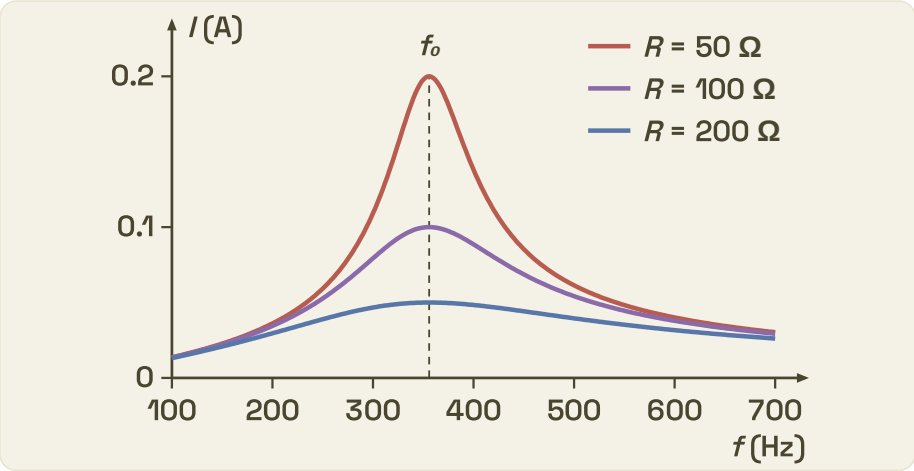


Figure 19.17 *Resonance curves for one coil and capacitor ($L = 0.1$ H, $C = 2$ μF, so $f_0 = 356$ Hz) on a 10 V rms supply, with three resistances, 50 Ω, 100 Ω and 200 Ω. All three peak at the same f_0; the smaller R, the taller and sharper the peak. Below f_0 the circuit is capacitive, above it inductive. The curves are not symmetrical: they fall away more slowly on the high-frequency side.*

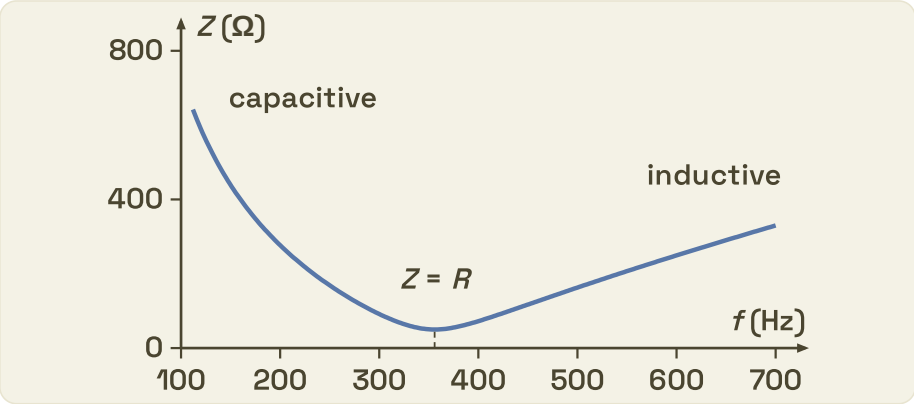


Figure 19.18 *The impedance of the same circuit ($R = 50$ Ω) against frequency. It is least, $Z = R$, at f_0, and rises on both sides: capacitive below, inductive above.*

Figure 19.18 shows the same story the other way up: the impedance dips to R at the resonant frequency and rises on either side, so the current in Figure 19.17 peaks where Z dips.

Check 9

In a series LCR circuit at resonance, which of these is **not** true?

a the impedance equals R
b the current and the e.m.f. are in phase
c the p.d. across the capacitor is zero
d the current has its greatest value

WORKED EXAMPLE

Example 19.8: the resistance of a solenoid's wire

A 50 V, 50 Hz a.c. supply is connected to a resistor of 40 Ω in series with a solenoid of inductance 0.2 H. The p.d. across the resistor is 20 V. What is the resistance of the solenoid's wire? (Take $\pi^2 = 10$.)

Answer

The current, from the resistor: $I = \dfrac{V_R}{R} = \dfrac{20}{40} = 0.5$ A.

The circuit's impedance: $Z = \dfrac{V}{I} = \dfrac{50}{0.5} = 100\ \Omega$.

Let the solenoid's wire have resistance r. The total resistance is $40 + r$, and

$$X_L^2 = (2\pi f L)^2 = 4\pi^2 f^2 L^2 = 4 \times 10 \times 2500 \times 0.04 = 4000\ \Omega^2$$

$$Z^2 = (40 + r)^2 + X_L^2 \quad \Rightarrow \quad 10\,000 = (40 + r)^2 + 4000$$

$$(40 + r)^2 = 6000, \quad 40 + r = 77.5, \quad r = 37.5\ \Omega$$

WORKED EXAMPLE

Example 19.9: a coil at two frequencies

A coil with inductance and resistance is connected to an oscillator giving a fixed output of 5 V rms. At 50 Hz the rms current is 1 A; at 100 Hz it is 0.625 A. (a) Find the inductance of the coil. (b) Find the ratio of the powers dissipated in the coil in the two cases.

Answer

(a) $Z_1 = \dfrac{5}{1} = 5\ \Omega$ at 50 Hz and $Z_2 = \dfrac{5}{0.625} = 8\ \Omega$ at 100 Hz.

$$Z_1^2 = R^2 + 4\pi^2 f_1^2 L^2 \qquad (i)$$

$$Z_2^2 = R^2 + 4\pi^2 f_2^2 L^2 \qquad (ii)$$

Subtracting (i) from (ii) removes R:

$$Z_2^2 - Z_1^2 = 4\pi^2 \left(f_2^2 - f_1^2\right) L^2$$

$$L^2 = \frac{8^2 - 5^2}{4\pi^2 \left(100^2 - 50^2\right)} = \frac{39}{4\pi^2 \times 7500} = 1.32 \times 10^{-4}\ \text{H}^2$$

$$L = \sqrt{1.32 \times 10^{-4}} = 0.0115\ \text{H} = 11.5\ \text{mH}$$

(And from (i), $R^2 = 25 - 4\pi^2 \times 2500 \times 1.32 \times 10^{-4} = 12$, so $R = 3.46\ \Omega$.)

(b) Only the resistance dissipates power, $P = I^2 R$, and R is the same in both cases:

$$\frac{P_1}{P_2} = \frac{I_1^2}{I_2^2} = \left(\frac{1}{0.625}\right)^2 = 2.56$$

CAREFUL

Don't forget the square on L

In Example 19.9 the subtraction gives L^2, not L: the reactance is $2\pi f L$, and it is squared in Z. Some solutions stop at 1.32×10^{-4} and call it the inductance; that is L^2. The inductance is its square root, 11.5 mH. A quick check: $X_L = 2\pi \times 50 \times 0.0115 = 3.6\ \Omega$, and $\sqrt{3.46^2 + 3.6^2} = 5\ \Omega$, as it should be.

WORKED EXAMPLE

Example 19.10: current, phase angle and power factor

An inductor of 80 mH is in series with a 20 Ω resistor across a 100 V (rms), 50 Hz a.c. source. Calculate (a) the rms current, (b) the phase angle between the current and the applied voltage, (c) the power factor.

Answer

$X_L = 2\pi f L = 2\pi \times 50 \times 80 \times 10^{-3} = 25.1\ \Omega$.

(a)

$$I_{\text{rms}} = \frac{V}{\sqrt{R^2 + X_L^2}} = \frac{100}{\sqrt{20^2 + 25.1^2}} = \frac{100}{32.1} = 3.1\ \text{A}$$

(b)

$$\tan\phi = \frac{X_L}{R} = \frac{25.1}{20} = 1.257, \qquad \phi = 51.5^\circ$$

The voltage leads the current by 51.5°.

(c) Power factor $= \cos\phi = \cos 51.5^\circ = 0.62$ (also $R/Z = 20/32.1 = 0.62$; power factor is explained in Section 19.13).

19.9 A.c. through a capacitance and a resistance (CR circuit)

Now connect a capacitance C and a resistance R in series across the a.c. source, $V = V_0 \sin\omega t$ (19.12). Again the same current flows through both, so take the current along OA:

- $V_R = IR$ is in phase with the current: along OA;
- $V_C = IX_C$ **lags** the current by 90°: straight **down** (AB).

The e.m.f. is their vector sum OB, and in the right-angled triangle OAB:

$$V^2 = V_R^2 + V_C^2 = (IR)^2 + (IX_C)^2 = I^2\left(R^2 + X_C^2\right)$$

$$I = \frac{V}{\sqrt{R^2 + X_C^2}} \qquad (19.13)$$

where $X_C = \dfrac{1}{\omega C} = \dfrac{1}{2\pi f C}$. The impedance of the CR circuit is

$$\boxed{Z = \sqrt{R^2 + X_C^2} = \sqrt{R^2 + \frac{1}{(2\pi f C)^2}}} \qquad (19.14)$$

and the e.m.f. now **lags behind** the current by the angle ϕ, where

$$\tan\phi = \frac{V_C}{V_R} = \frac{IX_C}{IR} = \frac{X_C}{R}, \qquad \boxed{\phi = \tan^{-1}\frac{X_C}{R}} \qquad (19.15)$$

(In the sign convention of Section 19.10, where φ is the angle by which the e.m.f. leads, this φ is negative: $\tan\phi = -X_C/R$.)

An LC circuit has a natural frequency too: a charged capacitor discharging through a coil makes the charge slosh back and forth between them, the energy swapping between the capacitor's electric field and the coil's magnetic field like a pendulum's between potential and kinetic energy. When the frequency of the applied e.m.f. matches this natural frequency, the current becomes as large as it can be, and the current and the e.m.f. are in phase. This is **electrical resonance**.

The resonant frequency

The current $I = V/Z$ is greatest when the impedance

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

is least. R doesn't change with frequency, and the bracket, being squared, can never be negative: Z is least when the bracket is zero,

$$\omega L - \frac{1}{\omega C} = 0 \quad \Rightarrow \quad \omega L = \frac{1}{\omega C} \quad \Rightarrow \quad \omega^2 = \frac{1}{LC} \quad \Rightarrow \quad \omega = \frac{1}{\sqrt{LC}}$$

If f_0 is the frequency of the source at which this happens, $\omega = 2\pi f_0$, so

$$\boxed{f_0 = \frac{1}{2\pi\sqrt{LC}}}$$

the **resonant (natural) frequency** of the series LCR circuit. At resonance:

- $X_L = X_C$, and V_L and V_C are equal and opposite, so they cancel;
- $\mathbf{Z = R}$, its least value, and the current is greatest: $I_{\max} = \dfrac{V}{R}$;
- $\phi = 0$: the current and e.m.f. are in phase; the circuit acts as a pure resistance.

Notice that R plays no part in f_0: it decides only how **tall** and how **sharp** the peak is.

The resonance curve

The current is $I = I_0 \sin\omega t$, with $I_0 = \dfrac{V_0}{Z}$. Leaving out the part that changes with time, $\sin\omega t$, the **amplitude** of the current is

$$I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{V_0}{\sqrt{R^2 + \left(2\pi f L - \frac{1}{2\pi f C}\right)^2}} \qquad (19.20)$$

A graph of this current against the frequency f is the **resonance curve**. It is small at low frequency (where X_C is huge), rises to a peak V_0/R at f_0, and falls again at high frequency (where X_L is huge). The smaller the resistance, the taller and narrower the peak.

2. $\omega L < \dfrac{1}{\omega C}$ ($X_L < X_C$): $\tan\phi$ and ϕ are negative: the **e.m.f. lags behind** the current. The circuit behaves like a CR circuit: **capacitive**.
3. $\omega L = \dfrac{1}{\omega C}$ ($X_L = X_C$): the two reactances cancel completely, $(X_L - X_C)^2 = 0$, so $\mathbf{Z = R}$ and $\phi = 0$. The circuit behaves as if it were a pure resistance: the current and e.m.f. are **in phase**, and the current is as large as it can be. This is **resonance** (Section 19.11).

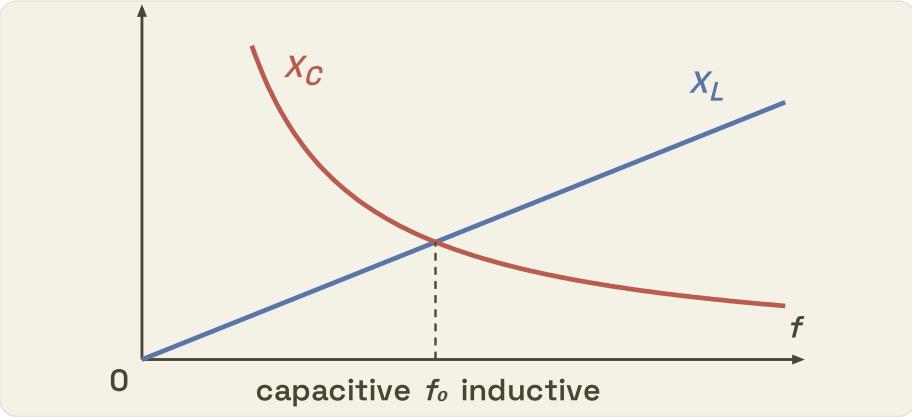


Figure 19.15 *How the two reactances of an LCR series circuit change with frequency. X_L (blue) rises in proportion to f; X_C (red) falls as $1/f$. Below the frequency f_0 where they cross, $X_C > X_L$ and the circuit is capacitive; above it, $X_L > X_C$ and it is inductive. At f_0 they are equal and cancel: series resonance.*

Check 8

In a series LCR circuit, $X_L = 30\ \Omega$, $X_C = 70\ \Omega$ and $R = 30\ \Omega$. The circuit is:

a inductive, with Z = 130 Ω
b capacitive, with Z = 50 Ω
c resistive, with Z = 30 Ω
d capacitive, with Z = 76 Ω

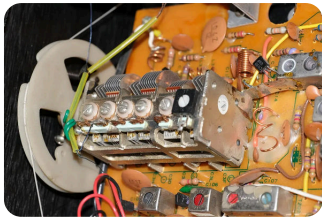


Figure 19.16 *A radio's variable capacitor. Turning the shaft swings one set of plates in between the other, changing the overlapping area and so the capacitance, and with it the frequency the radio is tuned to.*
Photo: Tyler C., CC0

19.11 Resonance in a series LCR circuit

Push a child on a swing in time with its natural swing, and small pushes build up a large swing. When a body that can oscillate is driven by an outside force at its **natural frequency**, it oscillates with the greatest amplitude: **resonance** 🕮 Grade 12.

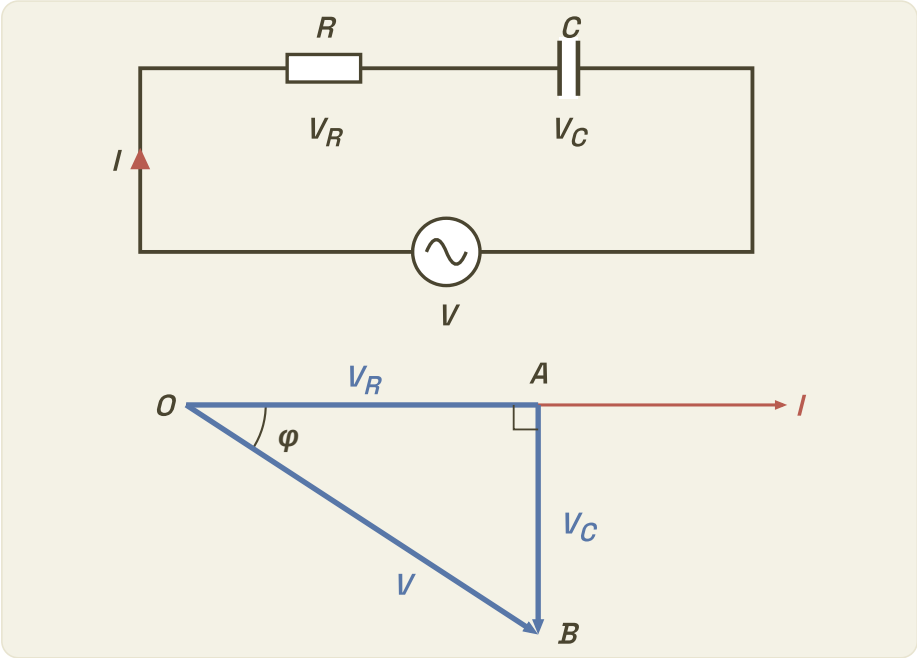


Figure 19.13 *A CR series circuit and its phasor diagram. V_R (OA) is in phase with the current I; V_C (AB) lags it by 90°, so it points down; the e.m.f. V (OB) is their vector sum, behind the current by the angle φ, with $\tan\phi = X_C/R$.*

WORKED EXAMPLE

Example 19.11: choosing the resistance

A CR series circuit has a 10 V (rms) source of frequency 1 kHz and a capacitance of 2 μF. What resistance R would let a current of 0.1 A flow?

Answer

$$Z = \frac{V}{I} = \frac{10}{0.1} = 100\ \Omega$$

$$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 10^3 \times 2 \times 10^{-6}} = 79.6\ \Omega, \qquad X_C^2 = 6332$$

$$R = \sqrt{Z^2 - X_C^2} = \sqrt{10\,000 - 6332} = \sqrt{3668} = 60.6\ \Omega$$

WORKED EXAMPLE

Example 19.12: when are the two reactances equal?

Calculate the reactance of an inductor of 100 mH and of a capacitor of 2 μF, both at 50 Hz. At what frequency are their reactances equal?

Answer

$$X_L = 2\pi f L = 2\pi \times 50 \times 0.1 = 31.4\ \Omega$$

$$X_C = \frac{1}{2\pi f C} = \frac{1}{2\pi \times 50 \times 2 \times 10^{-6}} = 1592\ \Omega$$

Raising the frequency raises X_L and lowers X_C, so they become equal at some frequency f_0:

$$2\pi f_0 L = \frac{1}{2\pi f_0 C} \quad \Rightarrow \quad f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.1 \times 2 \times 10^{-6}}} = 356\ \text{Hz}$$

(At f_0 both are 224 Ω. This is the **resonant frequency** of the pair, Section 19.11.)

WORKED EXAMPLE

Example 19.13: halving the current

A constant a.c. supply is connected across a resistance of 300 Ω in series with a capacitance of 6.67 μF; the frequency is $\frac{3000}{2\pi}$ Hz. Show how the current can be reduced to half its value by adding a resistance in series.

Answer

Here $\omega = 2\pi f = 3000$ rad/s, so

$$X_C = \frac{1}{\omega C} = \frac{1}{3000 \times 6.67 \times 10^{-6}} = 50\ \Omega$$

Let the added resistance be r. Before and after:

$$I = \frac{V}{\sqrt{R^2 + X_C^2}}, \qquad I' = \frac{V}{\sqrt{(R+r)^2 + X_C^2}}$$

For $I' = I/2$ the impedance must double: $\frac{I}{I'} = 2 = \sqrt{\frac{(R+r)^2 + X_C^2}{R^2 + X_C^2}}$, so

$$(R+r)^2 + X_C^2 = 4\left(R^2 + X_C^2\right) \quad \Rightarrow \quad (R+r)^2 = 4R^2 + 3X_C^2$$

$$(R+r)^2 = 4 \times 300^2 + 3 \times 50^2 = 360\,000 + 7500 = 367\,500$$

$$R + r = 606.2\ \Omega, \qquad r = 606.2 - 300 = 306.2\ \Omega$$

So a resistance of about 306 Ω added in series halves the current. (Not 300 Ω: doubling R alone would not double Z, because X_C stays.)

Check 7

A resistor of 30 Ω and a capacitor of reactance 40 Ω are in series on a 100 V a.c. supply. The current is:

a 3.3 A b 1.4 A

c 2.0 A d 2.5 A

19.10 A.c. through an inductance, a capacitance and a resistance (LCR circuit)

Now put all three in series: an inductance L, a capacitance C and a resistance R, across an a.c. source $V = V_0 \sin \omega t$ (19.16) of frequency f. The same current flows through all three; take it along OA. Then:

- $V_R = IR$ is in phase with the current: along OA;
- $V_L = IX_L$ leads the current by 90°: straight up;
- $V_C = IX_C$ lags the current by 90°: straight down.

V_L and V_C point in **exactly opposite** directions, so they partly cancel. If $X_L > X_C$, then $V_L > V_C$, and their resultant is $V_L - V_C$, upwards (AB). The supply's e.m.f. is the vector sum OB:

$$OB^2 = OA^2 + AB^2 \quad \Rightarrow \quad V^2 = V_R^2 + (V_L - V_C)^2 = (IR)^2 + (IX_L - IX_C)^2 = I^2\left[R^2 + (X_L - X_C)^2\right]$$

$$I = \frac{V}{\sqrt{R^2 + (X_L - X_C)^2}} \qquad (19.17)$$

The **impedance of the LCR circuit** is

$$\boxed{Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \qquad (19.18)$$

and the angle ϕ by which the e.m.f. **leads** the current is given by

$$\tan\phi = \frac{AB}{OA} = \frac{V_L - V_C}{V_R} = \frac{X_L - X_C}{R}, \qquad \boxed{\phi = \tan^{-1}\frac{X_L - X_C}{R}} \qquad (19.19)$$

The LR and CR circuits are special cases: put $X_C = 0$ (no capacitor; a capacitor shorted out) or $X_L = 0$ (no coil).

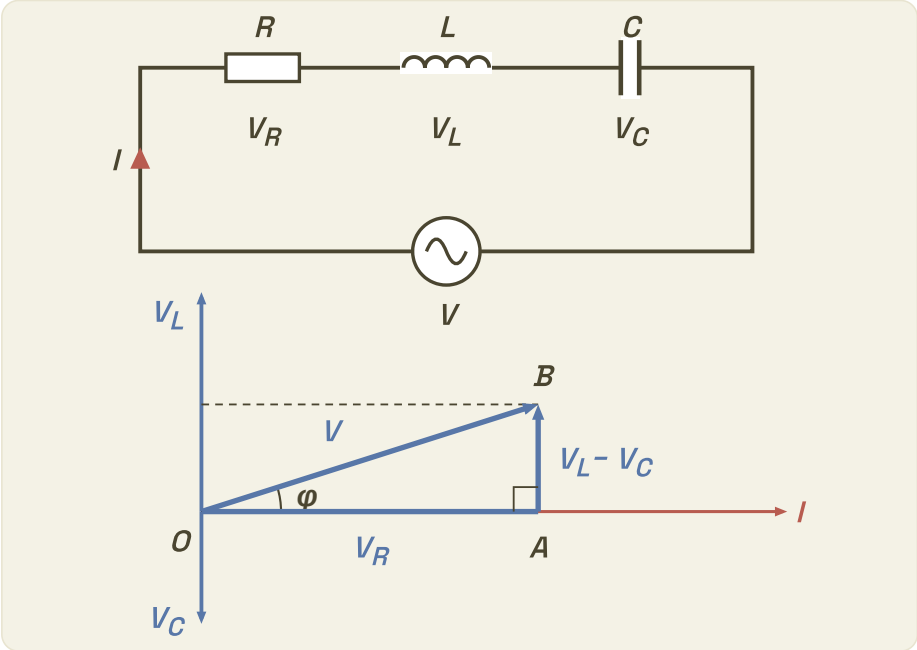


Figure 19.14 *An LCR series circuit and its phasor diagram, drawn for $X_L > X_C$. V_L points up and V_C down, so they partly cancel to $V_L - V_C$ (AB). The e.m.f. V (OB) is the vector sum of V_R and $V_L - V_C$, ahead of the current by φ.*

The phase angle depends on which reactance is larger. There are three cases:

1. $\omega L > \frac{1}{\omega C}$ ($X_L > X_C$): $\tan\phi$ is positive, so ϕ is positive: the **e.m.f. leads** the current. The circuit behaves like an LR circuit; we call it **inductive**.