Unit 16

Magnetic Fields

This unit online, with its interactives:
science-playground.com/s/12f16

Hold a compass under the cable of an electric kettle in Bhaktapur and switch the kettle on. Nothing happens: the cable carries two currents, in and out, side by side, and their magnetic effects cancel. Pull the two wires apart and lay one alone over the compass, and the needle swings the moment the current flows. That swing, first seen in 1820, is the start of this unit. Every motor, loudspeaker, hard disk, MRI scanner and electricity meter works because a current makes a magnetic field, and a magnetic field pushes on a current.

In this unit you will meet the magnetic field and its flux, find the force on a moving charge and on a wire, see why a coil turns in a field (and how that turning measures current), use the Hall effect to measure a field, and work out the field of a wire, a loop, a solenoid and a toroid, by the Biot–Savart law and by Ampère's law. It ends with two wires pulling on each other: the force that defined the ampere.



Figure 16.1 *Hans Christian Oersted (1777–1851), painted by C. A. Jensen in 1842.*
Photo: Christian Albrecht Jensen, Public domain

A STORY FROM SCIENCE

Oersted, and a needle that moved

Hans Christian Oersted was a Danish physicist and chemist. In April 1820, at the end of a lecture in Copenhagen, he passed a current from a battery through a wire held over a compass, and the needle turned. It was the first proof that **electricity and magnetism are connected**: a current makes a magnetic effect. Within weeks, scientists all over Europe were repeating the experiment, and within a few years Ampère, Biot, Savart and Faraday had worked out the laws in this unit and the next. This link, **electromagnetism**, led to the motor, the generator and the transformer, and to the technology of today. Oersted was a chemist too: he found **piperine**, the compound that makes black pepper hot, and in 1825 he was the first to separate the metal **aluminium** (a tiny lump of it, impure) from its compounds.

DID YOU KNOW?

Your data is stored as tiny magnets

A hard disk drive stores information magnetically. Its spinning platters are coated with a magnetic film divided into billions of tiny regions, each magnetized one way or the other: a 1 or a 0. A **write head**, a tiny electromagnet, sets the direction of each region as it passes; a **read head** senses the field of each region on the way back. Making and sensing magnetic fields this way, quickly and in very small places, is one of the ideas behind digital devices. (Phones and many laptops now use flash memory, which stores charge instead, but the world's data centers still keep most data on magnetic disks and tapes.)

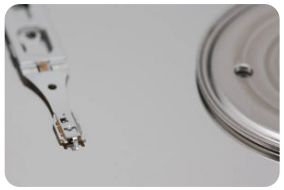


Figure 16.2 *Inside a hard disk drive: the arm carries the read and write head over the spinning platter, on a cushion of air thinner than a hair.*
Photo: Mfield , Matthew Field, http://www.photography.mattfield.com, CC BY-SA 3.0

16.1 Magnetic field and magnetic flux

How can a current, a flow of charge, turn a magnet that it never touches?

Oersted's discovery

Oersted's experiment showed the link between electricity and magnetism. When a current flows through a wire, it makes a **magnetic field** around the wire, and a compass needle placed near it is turned. Oersted found that:

- the needle turns only while the current flows;
- a **larger current** turns it **more**;
- **reversing the current** turns the needle the **other way**.

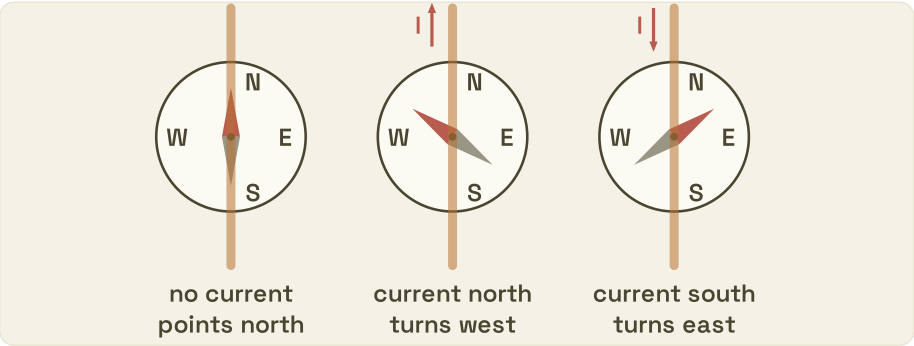


Figure 16.3 *Oersted's experiment. A wire runs north–south over a compass. With no current the needle points north along the wire; with a current the needle turns; with the current reversed it turns the other way.*

The experiment proved that electricity can make magnetism, and laid the ground for electromagnetism: the electric motor, the generator and the transformer all grew from it.

What Oersted's discovery did not explain

Oersted's discovery was a first step. It left a great deal for others to find:

1. It showed that a current makes a magnetic field, but not **how strong** the field is or how it changes with **distance** from the wire (the Biot–Savart law, Section 16.9).
2. It didn't show the full **shape and direction** of the field round the wire: circles round it, as Ampère and others showed (Section 16.2).
3. Oersted used a **steady** current, so his experiment couldn't show what happens when a current **changes**: that a changing magnetic field makes an e.m.f. (Faraday, 1831, the next unit).
4. It didn't show that electric and magnetic fields can travel together as **waves**: Maxwell showed that in the 1860s, and found that light is such a wave.
5. It didn't explain how a magnetic field **pushes on moving charges** (Section 16.3), which is what makes motors work.

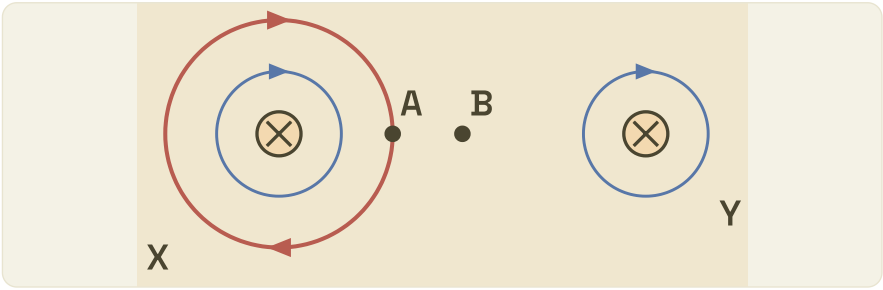


Figure 16.44 *For question 30: a card seen from above, with wires X and Y carrying currents down into it (⊗), a field line round each, the point A between them nearer X, and B midway. Drawn: the field line through A, a circle centered on X, clockwise (the answer to part a).*

A companion to your textbook, not a replacement for it: study from the book your school uses, and where the two differ, follow your teacher. Found a mistake? Please tell us.

25. A charged particle moving in a uniform magnetic field feels a force, the Lorentz force.
    a. What is the vector form of the Lorentz force?
    b. State Fleming's left-hand rule.
    c. A horizontal straight wire 5 cm long, of mass 1.2 g per meter, is placed at right angles to a uniform horizontal magnetic field of flux density 0.6 T. If the resistance of the wire is 3.8 Ω per meter, calculate the potential difference that must be applied between its ends to make it just self-supporting.

26. An electron is deflected by electric and magnetic fields.
    a. What path does an electron follow when it is projected at right angles to a uniform electric field?
    b. An electron passes through a region of space without being deflected. Does that mean there is no field there?
    c. Is there any condition in which an electron feels no force in a magnetic field?

27.
    a. State the Biot–Savart law in mathematical form.
    b. Find an expression for the magnetic field at the center of a circular coil of N turns carrying a current. Show the direction of the field in a suitable figure.

28.
    a. Two parallel conductors carrying currents in the same direction attract each other. Why? Explain by working out the force per unit length between them, with a suitable figure.
    b. Define one ampere from the force between two parallel conductors.

29. Two long, straight conductors PQ and XY are held a fixed distance apart. Each carries the same current in the same direction, out of the paper in the plan view from above in Figure 16.43.
    a. Draw arrows to show the direction of i. the magnetic field at Q due to the current in XY (label it B), ii. the force on Q due to the magnetic field of XY (label it F).
    b. i. State Newton's third law of motion. ii. Use it, and your answer to a ii, to state the direction of the force on XY.
    c. The magnetic flux density at a distance d from a long straight wire carrying a current I is $B = 2.0 \times 10^{-7} \times \frac{I}{d}$. Use this to explain why wires carrying alternating current are not normally seen to vibrate. Make reasonable estimates of the quantities involved.

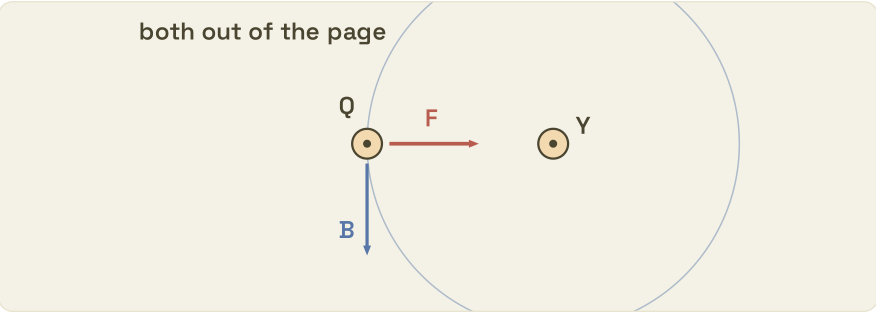


Figure 16.43 For question 29: a plan view from above, Q on the left and Y on the right, both currents out of the page (⊙). Drawn: the field of XY at Q, B, down the page, and the force on Q, F, towards Y (the answers to part a).

30. Figure 16.44 is a view from above of two wires X and Y. They carry equal currents vertically downward through a piece of card. One complete field line is drawn round each wire. Ignore the Earth's field.
    a. Draw the complete magnetic field line due to the current in X that passes through point A, and mark its direction.
    b. Point B is midway between the two wires. Explain why the magnetic field at B is zero.
    c. There is a force on wire Y due to the current in wire X. i. State its direction. ii. Explain why there is a force on wire Y.

Magnetic lines of force

A **magnetic line of force** (a **magnetic field line**) is an imaginary line drawn so that the **tangent** to it at any point gives the **direction of the magnetic field** there: the way a small compass needle placed there would point. An older definition says the same thing: it is the path along which a free, isolated north pole would move. (No isolated north pole has ever been found, but the idea gives the field's direction.)

The lines can be drawn with a compass. With the paper on the bar magnet, place a small compass near the N pole, mark the two ends of its needle with dots, move the compass on so that its tail is at the last dot, mark again, and so on: joining the dots gives one line.

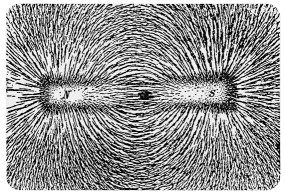


Figure 16.4 Iron filings on paper over a bar magnet. Each filing lines up with the field where it lies, tracing the lines of force out of one pole and round to the other.
Photo: Newton Henry Black, Public domain

The properties of magnetic lines of force:

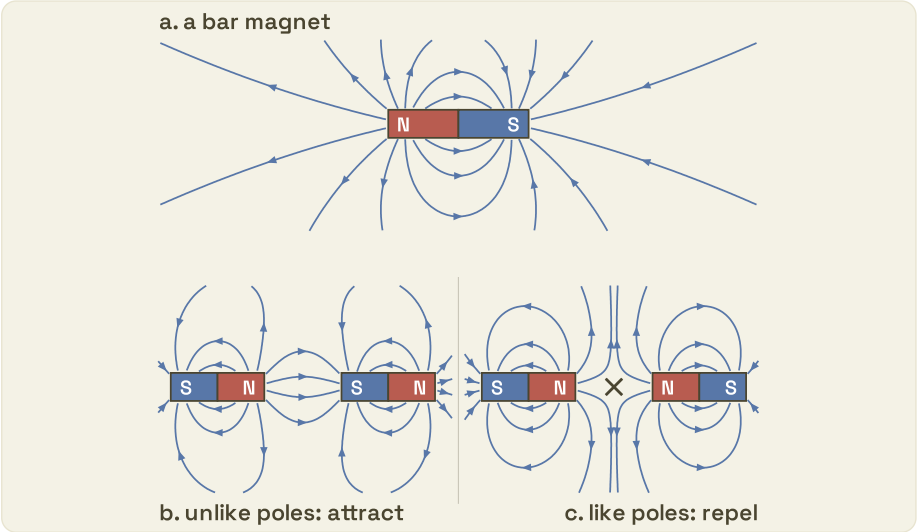


Figure 16.5 Lines of force worked out from the field. a. A bar magnet: the lines leave the N pole, curve round outside, and enter the S pole; inside the magnet they run from S to N, so every line is a closed loop. b. Unlike poles facing: lines run from one magnet into the other, and the magnets attract. c. Like poles facing (N to N): the lines from the two poles turn away from each other, and between them is a neutral point (×), where the two fields cancel.

1. Outside a magnet, they come out of the **north pole** and go into the **south pole**.
2. **Inside** the magnet, they run from the **south pole to the north pole**, so each line is a **closed loop** (magnetic field lines have no beginning or end).
3. They **never cross** each other. The tangent at a point gives the field's direction there; if two lines crossed, the field would have two directions at one point, which is impossible.
4. Where the lines are **closer together**, the field is **stronger**: the field's strength is in proportion to the number of lines crossing a unit area at right angles. So the lines are crowded near the poles and spread out farther away.
5. Lines of force running side by side in the same direction push each other apart sideways, and each line acts as if it were stretched, trying to shorten. This is a useful way of picturing forces: two N poles facing each other push apart, and an N and an S pull together.

Check 1
Why can two magnetic lines of force never cross?
a Because they repel each other.
b Because the field at a point has only one direction, the tangent to the line through it.
c Because they all start at the N pole.
d Because the field is zero where they would meet.

Magnetic flux

To talk about "how much field passes through a coil", we need a name for it. The **magnetic flux** through a plane surface is the product of the area and the component of the magnetic field at right angles to it:

$$\phi = BA \quad \text{... (16.1)}$$

when the field B is **normal** (at right angles) to the surface of area A. If the field makes an angle θ with the **normal** to the surface,

$$\phi = BA\cos\theta \quad \text{... (16.2)}$$

Only the field's component along the normal, $B\cos\theta$, passes through the surface. In vector form, $\phi = \vec{B}\cdot\vec{A}$, where $\vec{A}$ is a vector of size A along the normal. Flux is a **scalar**.

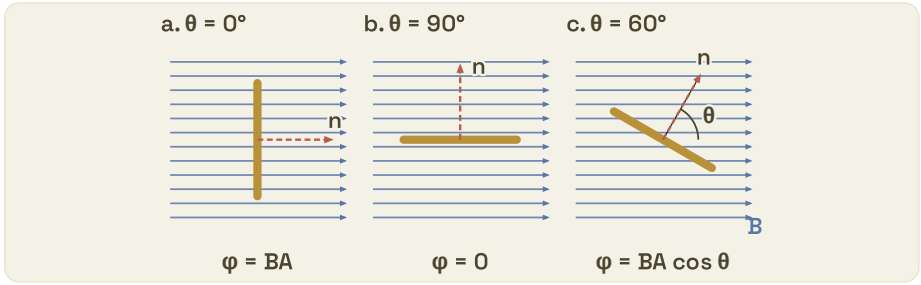


Figure 16.6 *The flux through a flat sheet. a. The field along the sheet's normal n (θ = 0): the most lines pass through, φ = BA. b. The field along the sheet (θ = 90°): no lines pass through, φ = 0. c. The normal at θ = 60° to the field: half as many lines pass through (cos 60° = ½), so φ = BA cos θ.*

You can picture flux as the **number of lines of force** passing through the surface: tilt the sheet, and fewer lines cross it.

Units of magnetic flux

In SI, the unit of magnetic flux is the **weber** (Wb). A uniform field of **1 tesla** at right angles to an area of **1 m²** gives a flux of **1 weber**. From $B = \phi/A$, the field is a flux per unit area, so

$$1\ \text{T} = 1\ \text{Wb/m}^2$$

and the field B is also called the **magnetic flux density**. In terms of other units, 1 Wb = 1 T m² = 1 N m/A (= 1 V s, as the next unit shows).

In the older CGS system, flux is measured in **maxwells** (Mx) and flux density in **gauss** (G), with 1 G = 1 Mx/cm². Since 1 T = 1 Wb/m² and 1 m² = 10^4 cm²,

$$1\ \text{T} = 10^4\ \text{G}, \qquad 1\ \text{Wb} = 10^8\ \text{Mx}$$

Flux linked with a coil

A coil of **N turns**, each of area A, placed at right angles to a uniform field B, has the flux BA through **each** turn. The total, called the **flux linkage**, is

$$N\phi = NBA \quad \text{... (16.3)}$$

and if the field makes an angle θ with the normal to the coil's plane,

$$N\phi = NBA\cos\theta \quad \text{... (16.4)}$$

Dimensions of magnetic flux

From the force on a moving charge (Section 16.3), $F = Bqv$, so $B = \dfrac{F}{qv} = \dfrac{F}{Itv}$ (since q = It). Then

$$[\phi] = [BA] = \frac{[F][A]}{[I][t][v]} = \frac{[\text{MLT}^{-2}][\text{L}^2]}{[\text{A}][\text{T}][\text{LT}^{-1}]} = [\text{ML}^2\text{T}^{-2}\text{A}^{-1}]$$

and the flux density B has dimensions $[\text{MT}^{-2}\text{A}^{-1}]$.

Check 2

Test yourself 16.1. A magnetic field is $\vec{B} = 5\hat{i} + 4\hat{j} + 9\hat{k}$ (in some unit). The magnetic flux through a surface of area 20 units lying in the y–z plane is:

a 100 units
b 20 units
c 80 units
d 180 units

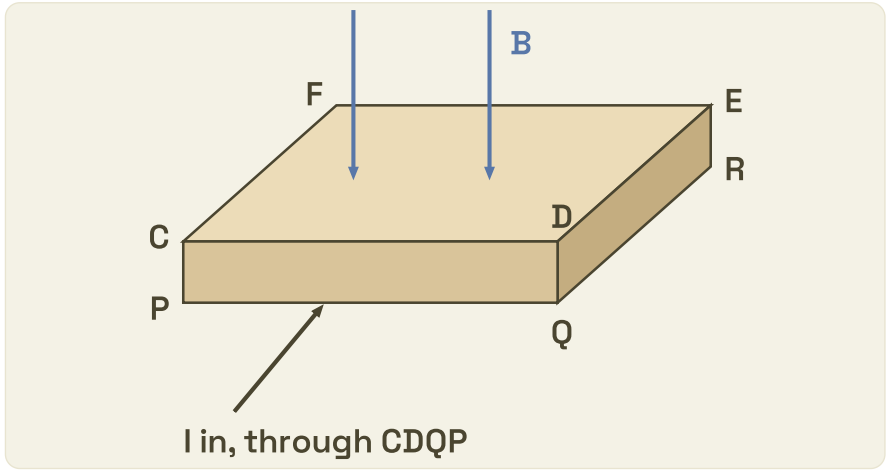


Figure 16.41 *For question 22: a thin slice CDEF (top) PQRS (bottom). The field B points down through the top face CDEF. The current I enters through the front face CDQP and leaves through the back face FERS.*

23. a. State the Hall effect.
A thin rectangular slice of aluminium has sides of 65 mm, 50 mm and 0.10 mm, as in Figure 16.42. A current I of 3.8 A is normal to the end face RSXY. In aluminium, the number of free electrons per unit volume is 6.0×10^{28} per m³. A uniform magnetic field of flux density 0.13 T is normal to the broad face QRYZ. A Hall voltage V_H is developed across the slice, given by $V_H = \dfrac{BI}{nqt}$.

b. Use the figure to state the distance t.

c. Calculate the Hall voltage.

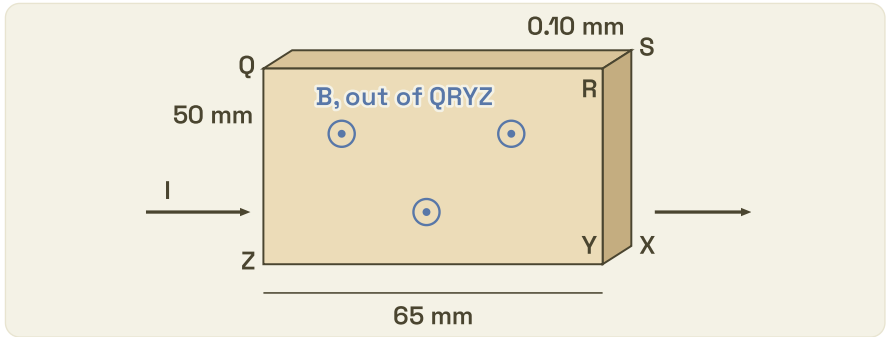


Figure 16.42 *For question 23: a thin aluminium slice, 65 mm long, 50 mm wide and 0.10 mm thick. The field B is at right angles to its broad face; the current I, 3.8 A, flows along its 65 mm length.*

24. a. State and explain the Biot–Savart law.

b. Use it to find the magnetic field at the center of a current-carrying circular coil.

c. A circular coil of 100 turns has a radius of 10 cm and carries a current of 5 A. Find the magnetic field at a point on its axis 5 cm from its center.

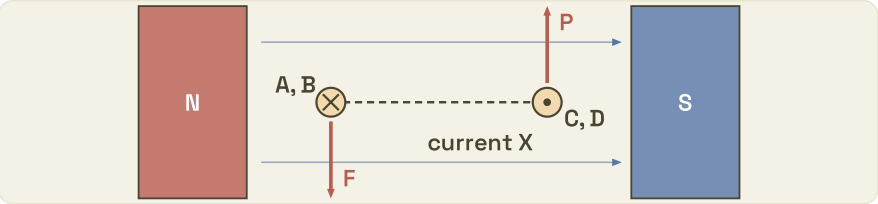


Figure 16.39 *For question 20: a rectangular coil ABCD between the N and S poles of a magnet, seen end-on, the field from N to S. In side AB, on the left, the current flows into the page (⊗), and the force F on AB is down the page. Drawn on side CD: the current X, out of the page (⊙), and the force P, up the page (the answers to part a).*

21. A proton moving with a constant velocity v enters a uniform magnetic field at right angles to its motion, as in Figure 16.40; the field is into the plane of the page. A uniform electric field is applied in the same region, so that the proton passes through undeflected.
    a. Draw an arrow labeled E on the figure to show the direction of the electric field.
    b. The proton is replaced by other particles; the electric and magnetic fields stay the same. State and explain the deflection, if any, of i. an α-particle with initial velocity v, ii. an electron with initial velocity 2v.

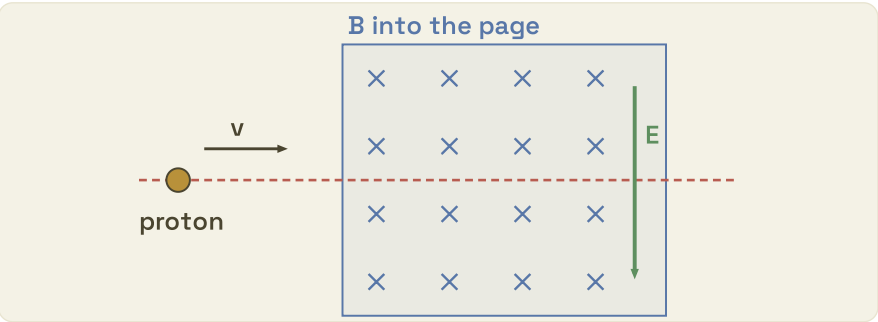


Figure 16.40 *For question 21: a proton moving to the right with velocity v enters a region of uniform magnetic field into the page. The arrow E (the answer to part a) points down the page.*

22. A thin slice of conducting material is placed at right angles to a uniform magnetic field of flux density B, as in Figure 16.41. The field is normal to faces CDEF and PQRS. A current I passes through the slice, normal to faces CDQP and FERS. A potential difference, the Hall voltage V_H, is developed across the slice.
    i. Name the faces between which the Hall voltage is developed.
    ii. The current is carried by charge carriers, each of charge +q, moving at speed v in the direction of the current; their number per unit volume is n. Derive an expression for the Hall voltage in terms of I, B, n, q and t, where t is the thickness of the slice.
    iii. Suggest why the Hall voltage is difficult to detect in a thin slice of copper.

Check 3
A coil of 50 turns, each of area 40 cm², is in a uniform field of 0.2 T. Its plane makes an angle of 30° with the field. The flux linkage is:
a 0.035 Wb
b 0.020 Wb
c 0.0004 Wb
d 0.040 Wb

Magnetic flux density

The **magnetic flux density** B at a point is the magnetic flux per unit area through a small area held **at right angles** to the field there:

$$B = \frac{\phi}{A}$$

It tells how concentrated the lines of force are. It is a **vector**, in the direction of the field, and it is what is usually called simply "the magnetic field". Its SI unit is the **tesla** (T); in CGS it is the gauss (1 T = 10^4 G). Section 16.3 gives it a second, equivalent definition, by the force it exerts on a moving charge.

A second quantity, the **magnetic field intensity** (magnetizing field) H, describes the field made by currents alone, whatever material is present. In a vacuum (or air) $B = \mu_0 H$. Its SI unit is the **ampere per meter** (A/m), and its CGS unit the **oersted** (Oe). Unit 17 uses it.

CAREFUL
B and H are different quantities

Some books call B the "magnetic field intensity" and give its unit as the oersted. The tesla and the gauss are units of **B**, the flux density; the ampere per meter and the oersted are units of **H**, the magnetizing field. In air, a field of 1 Oe has B = 1 G = 10^{-4} T, which is how the two get confused. Remember **1 T = 10^4 G**.

DID YOU KNOW?
How strong is a tesla?

The tesla is a large unit:

Field	Flux density
Earth's field at Kathmandu (2026)	about 49 μT (horizontal part about 35 μT)
a fridge magnet, at its surface	about 5 mT
a strong neodymium magnet, at its surface	about 1 T
an MRI scanner in a hospital	1.5 T or 3 T
the strongest steady field made in a laboratory (2025)	48.7 T
a neutron star (a magnetar)	about 10^{10} T

Several hospitals in Kathmandu have MRI scanners, whose 1.5 T is about 30,000 times the Earth's field.

Neutral points

A compass on a table feels two fields at once: the Earth's, whose **horizontal component** (the part along the table) points to magnetic north, and that of any magnet nearby. They add as vectors. Where they are equal and opposite, the resultant field is zero.

A **neutral point** is a point where the field of a magnet is equal and opposite to the horizontal component of the Earth's field, so that the resultant field there is zero. A compass placed at a neutral point doesn't point in any particular direction: it stays wherever it is left.

Where the neutral points are depends on how the magnet is placed in the magnetic meridian (the north–south line):

- **N pole pointing north:** to either side of the magnet's middle, the magnet's own field runs from its N end back to its S end, southward, against the Earth's field. The two neutral points lie **east and west**, on the magnet's **equatorial line** (the line through its middle at right angles to it).
- **N pole pointing south:** beyond the N end (south of the magnet) the magnet's field points away from the N pole, southward; beyond the S end (north of the magnet) it points into the S pole, southward again. On both sides it runs against the Earth's northward field, so the two neutral points lie **on the axial line**, beyond the two ends.

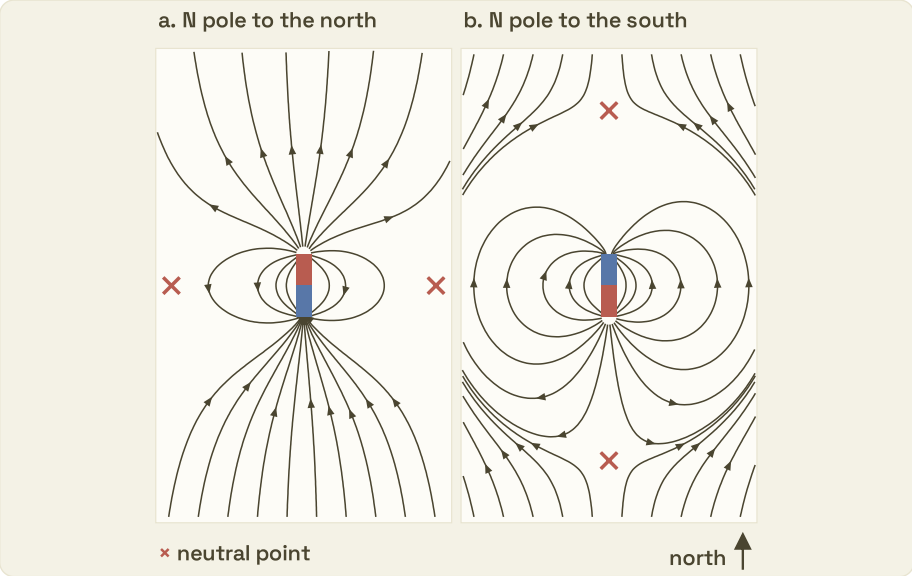


Figure 16.7 *The lines of force of a bar magnet on a table together with the Earth's field, worked out for a 6 cm magnet (its N pole red, its S pole blue), and its two neutral points (×). a. N pole pointing north: the neutral points are east and west of the magnet. b. N pole pointing south: they are on the magnet's axis, beyond its ends. Far from the magnet, in both, the lines run straight to the north.*

Check 4

A bar magnet lies on a table with its N pole pointing geographic south. A compass is placed at one of its neutral points. The compass:

a points north as usual
b points along the magnet, towards its N pole
c stays pointing in whatever direction it is left
d spins round continuously

16.2 Direction of the magnetic field round a current

What shape does the field of a straight wire have, and which way does it go round?

Three rules give the field's direction.

i. The right-hand thumb rule (for a straight conductor). Grasp the conductor in your **right hand** with the thumb pointing in the direction of the **current**. Your fingers then curl round the wire in the direction of the **magnetic lines of force**.

ii. Maxwell's corkscrew rule (for a straight conductor). Imagine a **right-handed corkscrew** (or an ordinary screw) being turned so that it moves forward along the conductor in the direction of the **current**. The direction in which its handle **turns** is the direction of the magnetic lines of force. A screw goes in when turned clockwise (seen from behind): so a current going away from you has a field going clockwise round it.

iii. The right-hand grip rule (for a circular coil or a solenoid). Curl the fingers of your **right hand** round the coil in the direction of the **current** in its turns. Your **thumb** then points in the direction of the magnetic field **along the coil's axis**: towards the coil's N pole.

+2q. Its change in mass is negligible. Find, in terms of v, the speed it must enter with to follow the same path through the field. Explain your working.

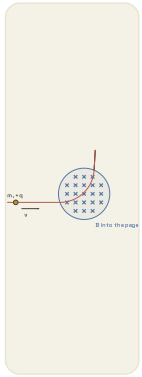


Figure 16.38 *For question 19: a particle (mass m, charge +q, speed v) heading for a circular region of uniform field into the page. Drawn: its path (the answer to b. i), an arc of a circle curving up the page inside the field, and a straight line beyond.*

20. Figure 16.39 shows a current-carrying coil in a magnetic field. The direction of the current in side AB of the coil is shown, and so is the force F on side AB.
    a. Draw an arrow labeled X on side CD to show the direction of the current in it, and an arrow labeled P to show the direction of the force on CD.
    b. Give two ways of increasing the forces F and P on the sides of the coil.
    c. i. Name the particles that flow in the metal wire of the coil.
    ii. The wire of the coil is replaced by a thinner wire of the same length and the same metal. State and explain how this changes the current in the coil.

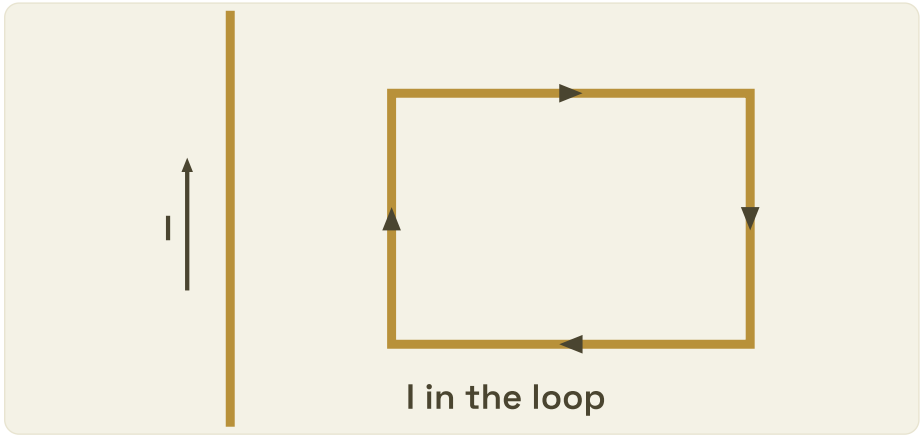


Figure 16.35 For question 16: a long straight wire with the current going up, and beside it, in the same plane, a rectangular loop whose near side also carries the current up and whose far side carries it down.

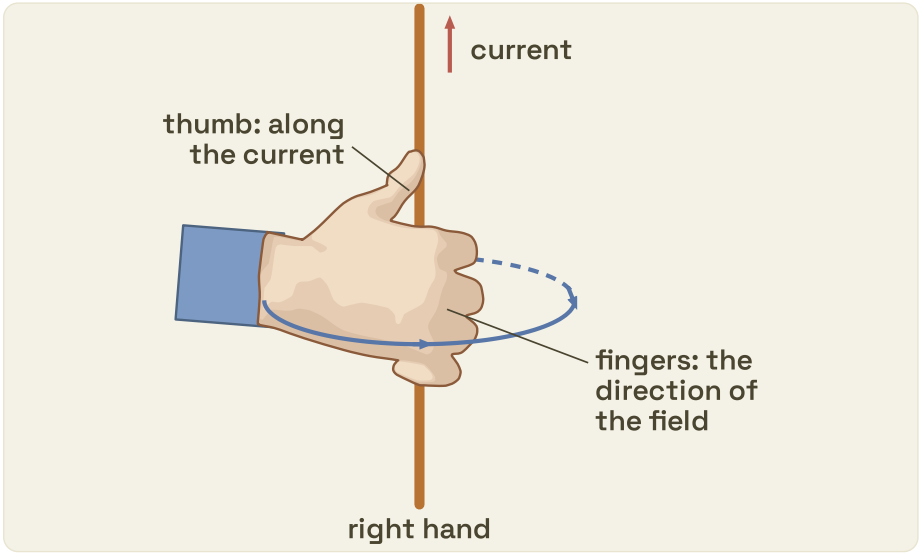


Figure 16.8 The right-hand thumb rule. The thumb points along the current; the curled fingers show the direction of the circular field lines. For a current going up, the field is anticlockwise when seen from above.

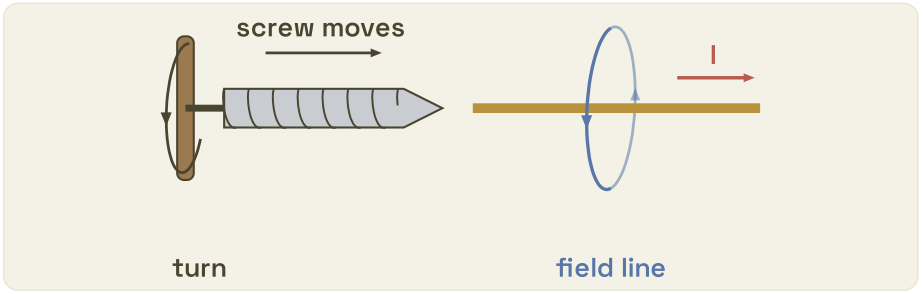


Figure 16.9 Maxwell's corkscrew rule. A screw driven along the wire in the direction of the current turns the way the field lines go round. Seen along the current (from behind the screw), both are clockwise.

Check 5

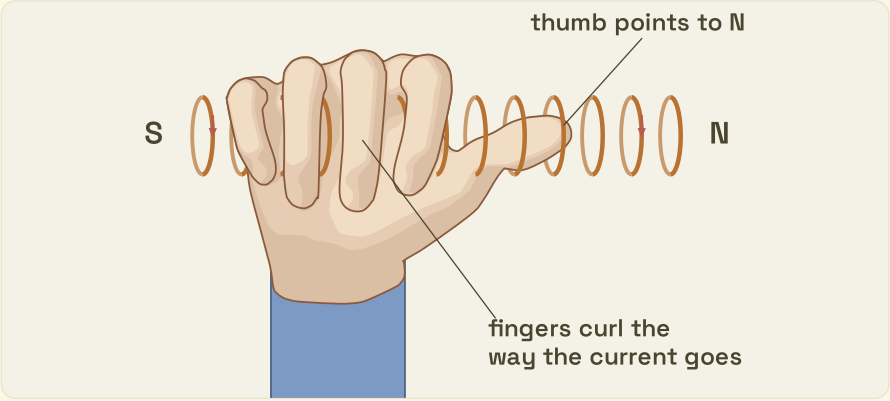


Figure 16.10 The right-hand grip rule for a solenoid: fingers along the current in the turns, thumb along the field inside, towards the north pole.

A long straight wire carries a current vertically **downward** through a horizontal card. Seen from above, the field lines round it on the card go:

a clockwise
b anticlockwise
c straight outward, away from the wire
d straight inward, towards the wire

16.3 Force on a moving charge: the Lorentz magnetic force

A magnetic field pushes on a current. But a current is only charges on the move. Does the field push on each moving charge?

It does. Take a charge +q moving with velocity $\vec{v}$ in a uniform magnetic field $\vec{B}$. Let the field point along the z-axis, and let the charge move in the y–z plane, its velocity at an angle θ to the field. Experiments show that the charge feels a force $\vec{F}$ along the **x-axis**: at right angles to both the velocity and the field. Its size is:

i. directly proportional to the charge: $F \propto q$ ... (16.5)
ii. directly proportional to the component of the velocity at right angles to the field: $F \propto v \sin\theta$ ... (16.6)
iii. directly proportional to the field: $F \propto B$ ... (16.7)

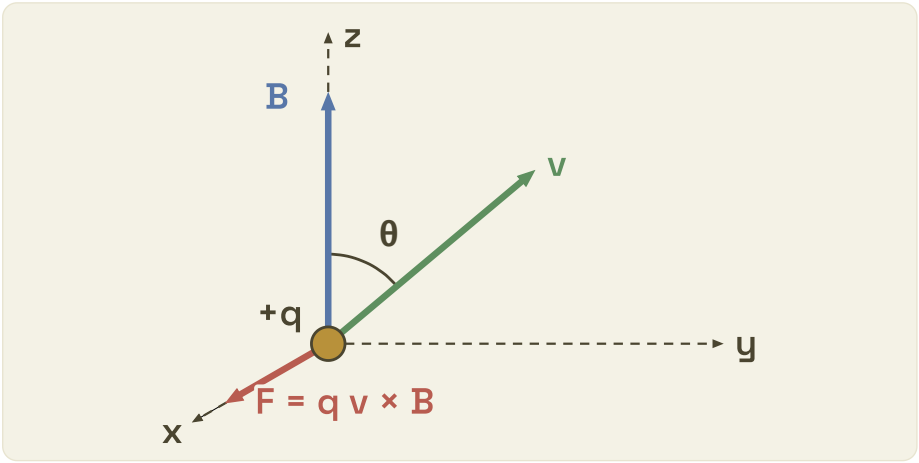


Figure 16.11 The magnetic force on a moving charge. The field B is along the z-axis and the charge +q moves with velocity v in the y–z plane, at angle θ to B. The force F is along the x-axis, at right angles to both v and B.

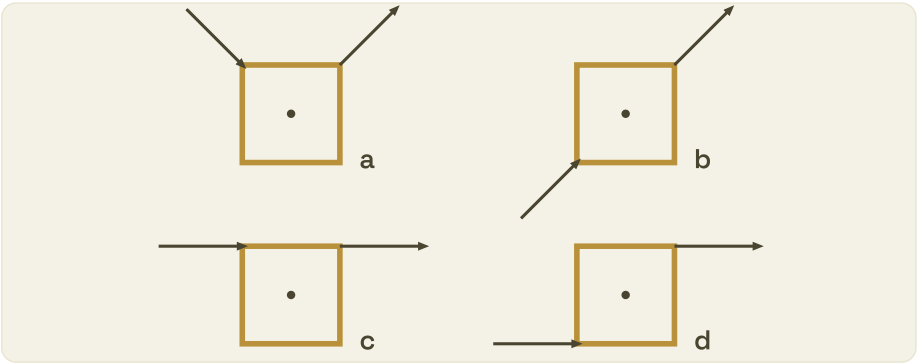


Figure 16.34 For question 13: four square frames of uniform wire, with the current led in and out in different ways. a. In and out at the two top corners, along the diagonals. b. In at the bottom-left corner and out at the top-right one, along the diagonal. c. In and out at the two top corners, along the line of the top side. d. In at the bottom-left corner along the line of the bottom side, out at the top-right corner along the line of the top side.

Check 23

14. In a moving-coil galvanometer, the deflection θ of the pointer is:

a (NA/kB) I
b (N/kAB) I
c (NAB/k) I
d (NAB/kI)

Check 24

15. Which one of the following statements about magnetic forces is correct?

a Magnetic forces always obey Newton's third law.
b Magnetic forces between moving charges do not always obey Newton's third law.
c Magnetic forces obey Newton's third law only for very large currents.
d Magnetic forces obey Newton's third law only in weak magnetic fields.

Check 25

16. A rectangular loop carrying a current I lies near a long straight wire, in the same plane, with one of its sides parallel to the wire. A steady current I is set up in the wire as in Figure 16.35. The loop will:

a turn about an axis parallel to the wire
b move away from the wire
c move towards the wire
d stay where it is

Check 18

9. A coil of area A_0 is in a magnetic field whose strength changes from B_0 to $4B_0$ in t seconds. The induced e.m.f. in the coil is:

a $4B_0/A_0t$ b $3B_0/A_0t$
c $3B_0A_0/t$ d $4B_0/A_0t$

Check 19

10. Magnetic lines of force:

a cannot intersect at all b intersect at infinity
c intersect within the magnet d intersect at the neutral point

Check 20

11. Two identical coils carrying equal currents have a common center, and their planes are at right angles. The ratio of the field at the center to the field of one coil alone is:

a 1 b $\sqrt{2}$
c $1/\sqrt{2}$ d 2

Check 21

12. A wire carrying 4 A runs in along a straight lead AB towards the center O of a circle of radius 3 cm, round three quarters of the circle, and out along a straight lead DE away from O, as in Figure 16.33. The magnetic field at O is:

a $(8\pi/3) \times 10^{-5}$ T b $(8\pi/3) \times 10^{-4}$ T
c $2\pi \times 10^{-5}$ T d $2\pi \times 10^{-4}$ T

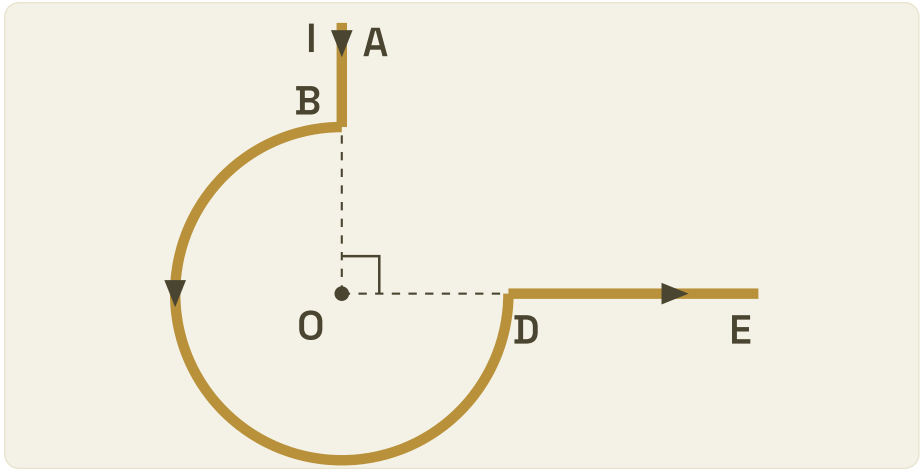


Figure 16.33 *For question 12: the current enters along AB and leaves along DE, both along radii of the circle; between B and D, the arc of three quarters of a circle; the gap BOD is a right angle.*

Check 22

13. A current flows through uniform square frames of wire as shown in Figure 16.34. In which case is the magnetic field at the center of the frame **not** zero?

a a b b
c c d d

Combining the three, $F \propto Bqv \sin\theta$, or $F = kBqv \sin\theta$. In SI the unit of B is chosen so that k = 1:

$$\boxed{F = Bqv \sin\theta} \quad \ldots (16.8)$$

In vector form, giving its direction as well,

$$\vec{F} = q\left(\vec{v} \times \vec{B}\right)$$

This is the **Lorentz magnetic force**. It is always at right angles to both $\vec{v}$ and $\vec{B}$. For a negative charge (an electron) the force is the other way.

Equation (16.8) also **defines the tesla**: a field of 1 T exerts a force of 1 N on a charge of 1 C moving at 1 m/s at right angles to it. So 1 T = 1 N/(C m/s) = **1 N/(A m)**.

Special cases:

- **i. Moving at right angles to the field** ($\theta = 90°$): $F = Bqv \sin 90^\circ = Bqv$, the **largest** force. As Section 16.4 shows, the charge then moves in a circle.
- **ii. Moving along the field** (parallel or antiparallel, $\theta = 0°$ or 180°): $F = Bqv \sin 0^\circ = 0$. The field has no effect on it.
- **iii. A charge at rest** (v = 0): $F = Bq(0) \sin\theta = 0$. A magnetic field exerts **no force on a charge at rest**. (A neutral particle, with q = 0, feels no magnetic force either.)

Check 6

An electron and a proton move side by side with the same velocity, at right angles to a uniform magnetic field. The magnetic forces on them are:

a equal in size and in the same direction b equal in size and in opposite directions
c different in size, because their masses differ d both zero

Fleming's left-hand rule

The direction of the force on a positive charge, or on a current, is given by **Fleming's left-hand rule**. Stretch the **thumb**, **forefinger** and **middle finger** of your **left hand** so that they are all at right angles to each other. Point the **forefinger** along the **field** (B) and the **middle finger** along the **motion of the positive charge**, the current (I or v). The **thumb** then points along the **force** (F).

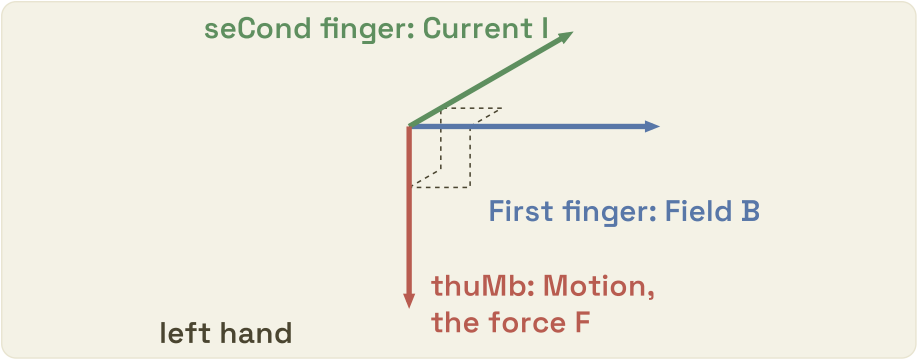


Figure 16.12 *Fleming's left-hand rule, as three directions at right angles: First finger along the Field, seCond finger along the Current (the motion of positive charge), thuMb along the Motion, the force. For an electron, point the second finger against its motion.*

A way to remember it: **F**irst finger, **F**ield; se**C**ond finger, **C**urrent; thu**M**b, **M**otion.

WORKED EXAMPLE

Example 16.1: the work done by a magnetic field

What is the work done by a magnetic field on a moving charge?

Answer

The magnetic force on the charge, $\vec{F} = q(\vec{v} \times \vec{B})$, is always at right angles to its velocity $\vec{v}$, and so at right angles to its displacement in every short time dt ($d\vec{s} = \vec{v}\,dt$). The work done in each step is

$$dW = F\,ds\cos 90^\circ = 0$$

so the total work done by the magnetic field is **zero**, whatever the path. The magnetic force can change the **direction** of the charge's velocity, but not its **speed** or its kinetic energy.

WORKED EXAMPLE

Example 16.2: an electron going north

An electron moves northward at 10^7 m/s in a uniform magnetic field of 3 T directed eastward. Find the force on it at that instant. (Charge of an electron = 1.6×10^{-19} C.)

Answer

q = 1.6×10^{-19} C, v = 10^7 m/s, B = 3 T; north and east are at right angles, so θ = 90°.

$$F = Bqv\sin\theta = 3 \times 1.6 \times 10^{-19} \times 10^7 \times \sin 90^\circ = \mathbf{4.8 \times 10^{-12}\ N}$$

Direction: for a positive charge, Fleming's rule with the field east and the motion north gives a force vertically **downward**. The electron is negative, so the force on it is vertically **upward**.

WORKED EXAMPLE

Example 16.3: a fast proton

A proton of kinetic energy 2 MeV moves vertically downward through a uniform horizontal magnetic field of 2.5 T. Find the force on it. (Mass of a proton = 1.7×10^{-27} kg, its charge = 1.6×10^{-19} C.)

Answer

First the speed. KE = 2 MeV = $2 \times 10^6 \times 1.6 \times 10^{-19}$ J = 3.2×10^{-13} J.

$$\frac{1}{2}mv^2 = \text{KE} \quad \Rightarrow \quad v = \sqrt{\frac{2\,\text{KE}}{m}} = \sqrt{\frac{2 \times 3.2 \times 10^{-13}}{1.7 \times 10^{-27}}} = 1.94 \times 10^7\ \text{m/s}$$

The motion (vertical) is at right angles to the field (horizontal), so θ = 90°:

$$F = Bqv = 2.5 \times 1.6 \times 10^{-19} \times 1.94 \times 10^7 = \mathbf{7.8 \times 10^{-12}\ N}$$

(about 8×10^{-12} N), horizontal and at right angles to the field.

YOUR TURN

Test yourself 16.5: at 30° to the field

An electron enters a magnetic field of flux density 3 T with a velocity of 10^6 m/s at an angle of 30° to the field. Find the magnitude of the force on it.

Answer

$$F = Bqv\sin\theta = 3 \times 1.6 \times 10^{-19} \times 10^6 \times \sin 30^\circ = 4.8 \times 10^{-13} \times 0.5 = \mathbf{2.4 \times 10^{-13}\ N}$$

YOUR TURN

Test yourself 16.6: crossing a field

Find the force on an electron crossing a uniform magnetic field of 0.3 T at right angles, with a velocity of 10^6 m/s. The charge of an electron is -1.6×10^{-19} C.

Answer

$$F = Bqv = 0.3 \times 1.6 \times 10^{-19} \times 10^6 = \mathbf{4.8 \times 10^{-14}\ N}$$

Exercise

Choose the correct option

Check 10

1. The work done in turning a small magnet of magnetic moment 10 A m² through 180° from the direction of a uniform magnetic field of 0.4×10^{-4} T is:

a 8×10^{-4} J
b zero
c 4×10^{-4} J
d none of these

Check 11

2. Points A and B lie on the extended axis of a bar magnet 2 cm long, at distances x and 2x (in cm) from the pole nearer to them. The ratio of the magnetic fields at A and B is:

a 4 : 1 exactly
b 8 : 1 exactly
c 4 : 1 approximately
d 8 : 1 approximately

Check 12

3. A toroidal solenoid has 300 turns and a mean radius of 10 cm. It has a soft-iron core of relative permeability 2000. The magnetic field in the core when a current of 1 A passes through it is:

a 0.12 T
b 0.24 T
c 1.2 T
d 12 T

Check 13

4. Two particles, each of mass m and charge q, are fixed to the two ends of a light rod of length 2l. The rod turns at a constant angular speed about an axis through its center, at right angles to it. The ratio of the magnetic moment of the system to its angular momentum is:

a q/m
b q/2m
c 2q/m
d q/πm

Check 14

5. The force per unit length between two straight conductors, each carrying the same current I, a distance b apart, is:

a $\mu_0 I/2\pi b^2$
b $\mu_0 I^2/2\pi b$
c $\mu_0 I/2\pi b$
d $\mu_0 I^2/2b$

Check 15

6. A coil of N turns, each of area A, carries a current I. The quantity NIA is its:

a magnetic flux
b magnetizing field
c magnetic susceptibility
d magnetic moment

Check 16

7. A long wire carrying a current I is bent at a right angle into two long straight arms. The magnetic field at a point inside the angle, a distance a from each arm, is:

a $\mu_0 I/4\pi a$
b $\mu_0 I/2a$
c $\mu_0 I(\sqrt{2}+1)/2\sqrt{2}\pi a$
d $\mu_0 I/2(a+\sqrt{a})$

Check 17

8. A coil of N turns and area A turns with angular velocity ω in a uniform magnetic field B. The maximum e.m.f. induced in it is:

a NABω
b ½NABω
c NAω/B
d Bω/NA

Answer these questions

17. A force acts on a wire carrying a current in a magnetic field. Figure 16.36 shows the direction of the current in the wire and

WORKED EXAMPLE

Example 16.27: where the force is zero

Two long parallel conductors carry 12 A and 8 A in the same direction, 10 cm apart. Where must a third parallel wire carrying a current be placed so that the force on it is zero?

Answer

With like currents, each of the two wires attracts the third. For the two pulls to cancel, the third wire must be **between** them, at a distance x from the 12 A wire, where the forces per unit length are equal:

$$\frac{\mu_0 \times 12 \times I}{2\pi x} = \frac{\mu_0 \times 8 \times I}{2\pi(0.1 - x)} \quad \Rightarrow \quad \frac{12}{x} = \frac{8}{0.1 - x}$$

$$12(0.1 - x) = 8x \quad \Rightarrow \quad 1.2 = 20x \quad \Rightarrow \quad x = \mathbf{0.06\ m}$$

6 cm from the 12 A wire (4 cm from the 8 A wire), whatever the third wire's current.

WORKED EXAMPLE

Example 16.28: two vertical wires

Two vertical conductors X and Y are 0.12 m apart and carry currents of 2 A and 4 A downward. Ignoring the Earth's field, find a. the distance from X of the point where the fields of X and Y cancel, b. the force per meter on X and on Y.

Answer

a. Between two like currents the fields point in opposite directions, so they cancel at a point P between the wires, a distance x from X, where

$$\frac{\mu_0 \times 2}{2\pi x} = \frac{\mu_0 \times 4}{2\pi(0.12 - x)} \quad \Rightarrow \quad 2(0.12 - x) = 4x \quad \Rightarrow \quad x = \mathbf{0.04\ m}$$

4 cm from X (nearer the smaller current).
b. The force per meter on X, in the field of Y:

$$\frac{F}{l} = \frac{\mu_0 I_X I_Y}{2\pi r} = \frac{2 \times 10^{-7} \times 2 \times 4}{0.12} = \mathbf{1.33 \times 10^{-5}\ N/m}$$

towards Y. The force per meter on Y is the same, **1.33 × 10⁻⁵ N/m**, towards X: the wires attract.

IN SHORT

Summary

- Oersted (1820): a current makes a magnetic field. Magnetic lines of force are closed loops, never cross, and are crowded where the field is strong. At a **neutral point** a magnet's field cancels the Earth's horizontal field.
- Flux: $\phi = BA\cos\theta$ (θ from the normal), in webers; flux linkage $N\phi$. Flux density $B = \phi/A$, in teslas: 1 T = 1 Wb/m² = 1 N/(A m) = 10^4 G.
- Directions: right-hand thumb rule and Maxwell's corkscrew rule (straight wire), right-hand grip rule (coil); Fleming's left-hand rule gives the force.
- Lorentz force: $F = Bqv\sin\theta$, $\vec{F} = q(\vec{v} \times \vec{B})$; it does no work. At right angles to B a charge moves in a circle, $r = mv/qB$, $T = 2\pi m/qB$ (the same for every speed); at an angle, in a helix.
- On a wire: $F = BIl\sin\theta$. On a coil: $\tau = NBIA\cos\alpha = MB\sin\theta$, M = NIA.
- Moving-coil galvanometer: radial field, $NBIA = k\theta$, so $I = (k/NBA)\theta$; current sensitivity NBA/k; voltage sensitivity NBA/kR.
- Hall effect: $V_H = BI/net$, large when n is small (semiconductors); used to measure B.
- Biot–Savart law: $dB = \frac{\mu_0}{4\pi}\frac{I\,dl\sin\theta}{r^2}$. Center of a coil $\mu_0 NI/2r$; on its axis $\mu_0 NIa^2/2(a^2 + x^2)^{3/2}$; straight wire $\frac{\mu_0 I}{4\pi a}(\sin\varphi_1 + \sin\varphi_2)$, long wire $\mu_0 I/2\pi a$.
- Ampère's law: $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enclosed}}$. Solenoid $\mu_0 nI$; toroid $\mu_0 NI/2\pi r$, zero outside.
- Parallel wires: $F/l = \mu_0 I_1 I_2/2\pi r$; like currents attract, unlike repel. 1 A gives 2×10^{-7} N/m at 1 m.

16.4 Motion of a charged particle in a uniform magnetic field

If a magnetic field can never speed a charge up or slow it down, what can it do?

It can only **turn** it. What path the charge takes depends on the angle between its velocity and the field.

WORKED EXAMPLE

Example 16.4: largest and smallest force

Does a charge moving in a uniform magnetic field always feel a force? If so, what are the largest and smallest values of the force?

Answer

Not always. A charge +q moving with velocity $\vec{v}$ at angle θ to a uniform field $\vec{B}$ feels the force $\vec{F} = q(\vec{v} \times \vec{B})$, of size $F = Bqv\sin\theta$, at right angles to both.

- It is **least**, $F_{\min} = 0$, when sin θ = 0: θ = 0° or 180°, moving along the field.
- It is **greatest**, $F_{\max} = Bqv$, when sin θ = 1: θ = 90°, moving at right angles to the field.

Moving at right angles to the field: a circle

Let a particle of mass m and charge q move with speed v at right angles to a uniform field B. The force Bqv is always at right angles to its velocity, and the same size everywhere: exactly what is needed for **uniform circular motion**. The magnetic force provides the centripetal force:

$$Bqv = \frac{mv^2}{r} \quad \Rightarrow \quad \boxed{r = \frac{mv}{qB}}$$

A faster particle, or a heavier one, moves in a bigger circle; a stronger field bends it more tightly. The time for one round, the **period**, is

$$T = \frac{2\pi r}{v} = \frac{2\pi m}{qB}, \qquad f = \frac{1}{T} = \frac{qB}{2\pi m}$$

The speed has canceled out: in a given field, a particle takes the **same time for each circle whatever its speed**. A faster one goes round a bigger circle in the same time. This frequency is the **cyclotron frequency**: the cyclotron, a particle accelerator, gives particles a push at that steady beat, and they spiral outward as they speed up.

At an angle to the field: a helix

If the velocity makes an angle θ with the field, split it into two parts:

- $v\cos\theta$ **along** the field, which feels no force and stays the same;
- $v\sin\theta$ **at right angles** to it, which is turned round in a circle of radius $r = \frac{mv\sin\theta}{qB}$, in the same period $T = \frac{2\pi m}{qB}$.

Together they make a **helix** (a spiral like a spring) round the field lines. In each period it moves along the field a distance called the **pitch**:

$$p = (v\cos\theta)\,T = \frac{2\pi mv\cos\theta}{qB}$$

WORKED EXAMPLE

Example 16.5: three paths

What is the path of a charged particle in a uniform magnetic field if it moves i. parallel to the field, ii. at right angles to the field, iii. at an angle θ to the field?

Answer

i. **A straight line.** Along the field sin θ = 0, so there is no force: the particle keeps its velocity.
ii. **A circle.** The force Bqv is always at right angles to the velocity and constant in size, so it provides a centripetal force: a circle of radius $r = mv/(qB)$, in the plane at right angles to the field.
iii. **A helix.** The part of the velocity along the field, $v\cos\theta$, is unchanged; the part at right angles, $v\sin\theta$, goes round a circle. Together: a helix whose axis is along the field.

WORKED EXAMPLE

Example 16.6: no work, but a change

If a magnetic force does no work on a charged particle, how can it affect its motion?

Answer

Work changes a particle's **speed** (its kinetic energy); a force can also change the **direction** of its velocity. The magnetic force $F = Bqv\sin\theta$ is always at right angles to the velocity, and so to the displacement: its work, $W = Fs\cos 90^\circ$, is zero, and the speed stays the same. But because it is at right angles to the motion, it keeps **turning** the velocity, as a string turns a stone whirled round. So the particle moves in a circle (or a helix) at a steady speed: the field changes its velocity without changing its speed.

YOUR TURN

A proton in a circle

A proton (mass 1.67×10^{-27} kg, charge 1.6×10^{-19} C) moves at 3×10^{6} m/s at right angles to a uniform field of 0.5 T. Find the radius of its circle and the time for one round. How would they change if its speed were doubled?

Answer

$$r = \frac{mv}{qB} = \frac{1.67\times10^{-27}\times3\times10^{6}}{1.6\times10^{-19}\times0.5} = \mathbf{0.063\ m}\ (6.3\ \text{cm})$$

$$T = \frac{2\pi m}{qB} = \frac{2\pi\times1.67\times10^{-27}}{1.6\times10^{-19}\times0.5} = \mathbf{1.3\times10^{-7}\ s}$$

At double the speed the radius doubles, to 12.5 cm, and the period stays the **same**: twice the distance at twice the speed.

Crossed electric and magnetic fields: the velocity selector

Put an electric field E and a magnetic field B at right angles to each other and to a beam of charged particles, arranged so that the two forces on a particle are opposite. The electric force qE is the same for every particle; the magnetic force qvB depends on its speed. Only particles for which

$$qE = qvB \qquad \Rightarrow \qquad v = \frac{E}{B}$$

pass straight through. Slower ones are pushed one way, faster ones the other. The condition doesn't depend on the charge or the mass: this **velocity selector** picks out particles of one speed, whatever they are. J. J. Thomson used it in 1897 to measure the electron's charge-to-mass ratio, the start of Unit 19.

DID YOU KNOW?

The Earth's magnetic shield

Charged particles streaming from the Sun meet the Earth's magnetic field and are turned aside. Those that get trapped spiral along the field lines, in helices, bouncing between the north and south polar regions, in two belts named after James Van Allen, who found them with the first American satellite in 1958. Near the poles, where the field lines dip into the atmosphere, some spiral down and strike the air 100 km and more up, making it glow: the **aurora**. Nepal is too far south to see it, except in the very rare great storms: in May 2024, a faint red aurora was photographed low in the northern sky from Ladakh, in India.

16.5 Force on a current-carrying conductor in a magnetic field

The current in a wire is a stream of free electrons drifting along it. In a magnetic field, each feels a Lorentz force; the wire passes the push on to itself (Test yourself 16.3). Adding up the forces on all the electrons gives the force on the wire.

Take a straight conductor of length l and cross-sectional area A, carrying a current I, at an angle θ to a uniform field B. Let it have n free electrons per unit volume, each of charge e, drifting with speed v_d.

Step 1. Each electron feels a force $f = Bev_d\sin\theta$.

Step 2. The wire holds $N = nAl$ free electrons (its volume times n), so the total force is

$$F = (nAl)\,Bev_d\sin\theta = B\,(nAev_d)\,l\sin\theta$$

Step 3. But the current is $I = nAev_d$ (Grade 11, Unit 23). So

$$\boxed{F = BIl\sin\theta} \quad \ldots (16.9)$$

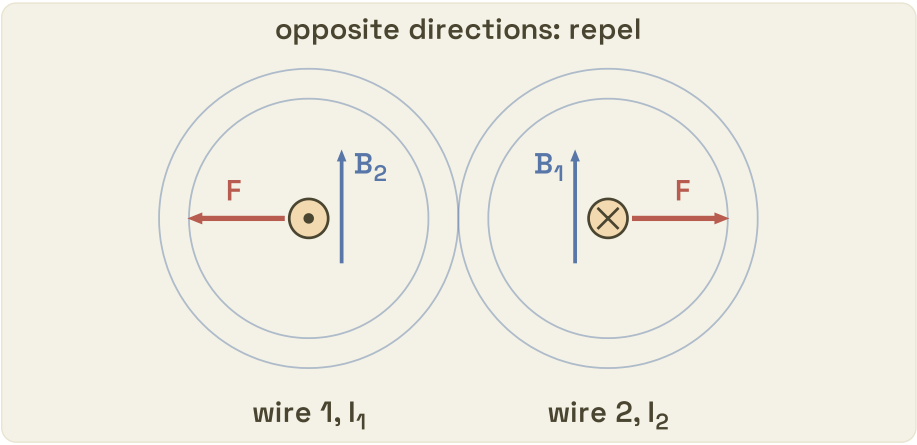


Figure 16.32 *Two parallel wires carrying currents in opposite directions (wire 1 out of the page, wire 2 into it). Each wire's field at the other is reversed from the figure above, and the wires push each other apart.*

Like currents attract; unlike currents repel. (The opposite of the rule for charges.)

DID YOU KNOW?

Wires of different lengths

If one of the two parallel wires is shorter, the force on the long wire and the force on the short one are still equal and opposite: each is the force per unit length, $\mu_0 I_1 I_2/2\pi r$, times the length of the **short** wire, the length over which the two run side by side (as long as the long wire stretches well past both ends of the short one).

Defining the ampere

Take two long, thin, straight wires 1 m apart in a vacuum, each carrying 1 A. The force per meter between them is

$$\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi r} = \frac{4\pi\times10^{-7}\times1\times1}{2\pi\times1} = 2\times10^{-7}\ \text{N/m}$$

So **one ampere** is the steady current which, flowing in each of two infinitely long, straight, parallel conductors of negligible cross-section placed 1 m apart in a vacuum, produces between them a force of **2×10^{-7} newton per meter** of length.

DID YOU KNOW?

The ampere since 2019

This was the official definition of the ampere from 1948 until 2019. Since 20 May 2019, the SI defines the ampere by fixing the charge of an electron, e = 1.602 176 634 × 10^{-19} C exactly: one ampere is a flow of 1/(1.602 176 634 × 10^{-19}) elementary charges per second. μ_0 is now a measured number, 1.256 637 061 × 10^{-6} N/A², which agrees with $4\pi\times10^{-7}$ to better than one part in a billion: the force between two wires carrying 1 A, 1 m apart, is still 2×10^{-7} N/m.

WORKED EXAMPLE

Example 16.26: a solenoid contracts

A current-carrying solenoid tends to contract. Explain why.

Answer

A solenoid is a row of turns side by side, all carrying the current in the **same direction**. Neighboring turns act like parallel conductors with like currents, and the force between them, $\mu_0 I_1 I_2/2\pi r$ per unit length, is **attractive**. Every turn pulls on its neighbors, so the solenoid tends to shorten. (In very strong electromagnets, such as those of an MRI scanner, these forces are large, and the windings must be clamped firmly.)

Check 9
A toroid of 500 turns has an inner radius of 9 cm and an outer radius of 11 cm, and carries 2 A. The field inside its windings, at r = 10 cm, is:
a 1 mT
b 2 mT
c 0
d 20 mT

16.11 Force between parallel current-carrying conductors

Each wire of a pair makes a field, and the other wire sits in it. So what do two wires do to each other?

A current-carrying conductor makes a magnetic field round it, and a current-carrying conductor in a magnetic field feels a force. So two current-carrying wires side by side push or pull on each other. Ampère discovered this within weeks of hearing of Oersted's experiment, in 1820.

a. Currents in the same direction

Take two long, straight, parallel conductors 1 and 2, a distance r apart, carrying currents I_1 and I_2 in the **same direction**.

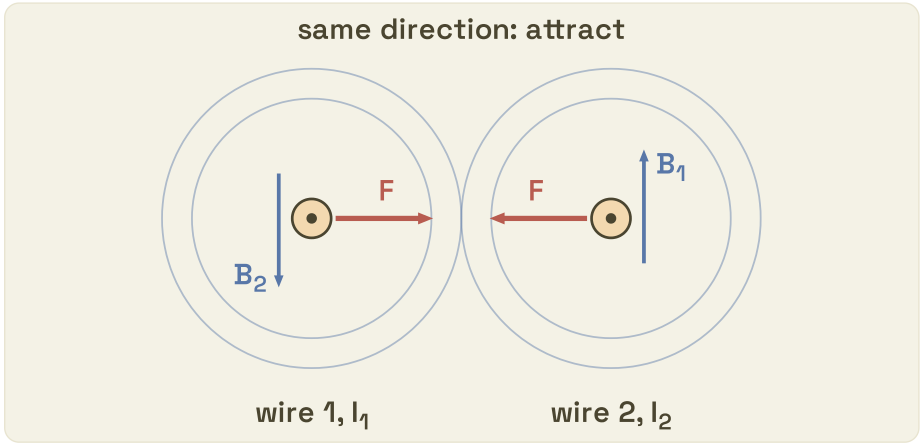


Figure 16.31 *Two parallel wires carrying currents in the same direction (seen end on, both out of the page). The field of wire 1 at wire 2, B_1, and the field of wire 2 at wire 1, B_2, are drawn. By Fleming's left-hand rule, each wire is pulled towards the other.*

Step 1. The field of wire 1 at the position of wire 2 is

$$B_1 = \frac{\mu_0 I_1}{2\pi r}$$

at right angles to wire 2 (by the right-hand thumb rule).
Step 2. A length l of wire 2, carrying I_2 at right angles to this field, feels a force

$$F_{21} = B_1 I_2 l \sin 90^\circ = \frac{\mu_0 I_1 I_2 l}{2\pi r} \quad \Rightarrow \quad \frac{F_{21}}{l} = \frac{\mu_0 I_1 I_2}{2\pi r} \quad \ldots (16.19)$$

By Fleming's left-hand rule, it is directed **towards wire 1**.
Step 3. In the same way, the field of wire 2 at wire 1, $B_2 = \mu_0 I_2 / 2\pi r$, gives a force per unit length on wire 1

$$\frac{F_{12}}{l} = \frac{\mu_0 I_1 I_2}{2\pi r} \quad \ldots (16.20)$$

directed **towards wire 2**.

The two forces per unit length are equal and opposite, and the wires **attract** each other.

b. Currents in opposite directions

If the currents are in **opposite** directions, the field of each wire at the other is reversed, and so is each force. The sizes are the same,

$$\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi r} \quad \ldots (16.21)$$

but now each wire is pushed **away** from the other: the wires **repel** each other.

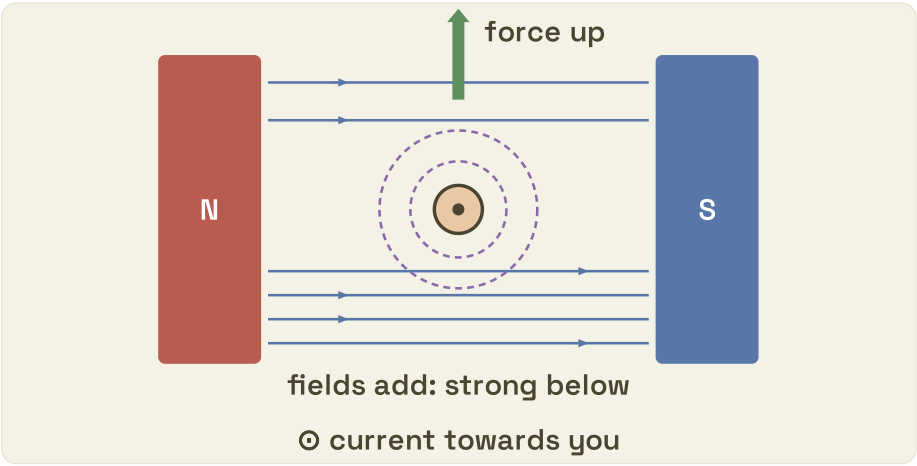


Figure 16.13 *The motor effect. A wire carrying a current between the poles of a magnet feels a force at right angles to both the current and the field, and moves.*

In vector form, $\vec{F} = I\left(\vec{l} \times \vec{B}\right)$, where $\vec{l}$ has the wire's length and points along the current. The force is at right angles to both the wire and the field, as Fleming's left-hand rule gives. This holds when the field is **uniform** along the whole length of the wire.

Special cases:

- **Case I.** The wire **along the field** ($\theta = 0^\circ$ or 180°): $\sin\theta = 0$, so $\mathbf{F = 0}$. A wire parallel to the field feels no force.
- **Case II.** The wire **at right angles** to the field ($\theta = 90^\circ$): $\sin\theta = 1$, so $\mathbf{F = BIl}$, the largest force.

Check 7
A straight wire carrying a current lies along the lines of a uniform magnetic field. The force on it:
a is zero
b is BIl, along the wire
c is BIl, at right angles to the wire
d depends on the direction of the current

WORKED EXAMPLE
Example 16.7: a wire at 60°

A wire 2 m long carrying a current of 10 A is placed in a field of flux density 0.34 T. What is the force on the wire if it is at 60° to the field?

Answer
l = 2 m, B = 0.34 T, I = 10 A, $\theta = 60^\circ$.

$$F = BIl\sin\theta = 0.34 \times 10 \times 2 \times \sin 60^\circ = 6.8 \times 0.866 = \mathbf{5.89\ N}$$

WORKED EXAMPLE
Example 16.8: the force on one electron

A copper wire has 1×10^{29} free electrons per cubic meter and a cross-sectional area of 2 mm², and carries a current of 6 A. Find the force on each electron if the wire is placed at right angles to a uniform field of 0.1 T.

Answer
$n = 1 \times 10^{29}$ /m³, $A = 2\ \text{mm}^2 = 2 \times 10^{-6}\ \text{m}^2$, I = 6 A, B = 0.1 T.
The drift speed: $v_d = \frac{I}{neA}$. The force on one electron:

$$f = Bev_d = Be\frac{I}{neA} = \frac{BI}{nA} = \frac{0.1 \times 6}{1 \times 10^{29} \times 2 \times 10^{-6}} = \mathbf{3 \times 10^{-24}\ N}$$

A tiny force on each, but there are 2×10^{23} free electrons in each meter of the wire: together, 0.6 N per meter (= BIl).

WORKED EXAMPLE

Example 16.9: halving the force

A vertical straight conductor X, 0.5 m long, is in a uniform horizontal magnetic field of 0.1 T. i. Find the force on X when a current of 4 A passes through it. ii. Through what angle must X be turned, in a vertical plane, so that the force on it is halved?

Answer

i. The wire is vertical and the field horizontal, so θ = 90°:

$$F = BIl\sin 90^\circ = 0.1 \times 4 \times 0.5 \times 1 = \mathbf{0.2\ N}$$

ii. The force is to be F′ = 0.1 N, so

$$\sin\theta' = \frac{F'}{BIl} = \frac{0.1}{0.1 \times 4 \times 0.5} = 0.5 \quad \Rightarrow \quad \theta' = 30^\circ$$

The wire must make 30° with the field. It started at 90° to the field, so it must be turned through **60°** (in the vertical plane that contains the field).

WORKED EXAMPLE

Example 16.10: balancing a rod's weight

A straight horizontal rod X, of mass 50 g and length 0.5 m, is in a uniform magnetic field of 0.2 T at right angles to it. Find the current in X if the magnetic force on it just balances its weight. (g = 10 N/kg)

Answer

m = 50 g = 0.05 kg, l = 0.5 m, B = 0.2 T, θ = 90°. Weight W = mg = 0.05 × 10 = 0.5 N.
For balance, the magnetic force (upward) equals the weight:

$$BIl = mg \quad \Rightarrow \quad I = \frac{mg}{Bl} = \frac{0.5}{0.2 \times 0.5} = \mathbf{5\ A}$$

The current must flow the way that makes the force upward (Fleming's left-hand rule).

WORKED EXAMPLE

Example 16.11: a self-supporting wire

A horizontal straight wire 5 cm long, of mass 1.2 g per meter, is placed at right angles to a uniform horizontal magnetic field of 0.6 T. The resistance of the wire is 3.8 Ω per meter. Find the potential difference that must be applied between its ends to make it just self-supporting.

Answer

l = 5 cm = 0.05 m, B = 0.6 T, θ = 90°.
Resistance of the wire: R = 3.8 Ω/m × 0.05 m = 0.19 Ω.
Mass of the wire: m = 1.2 g/m × 0.05 m = 0.06 g = 6 × 10⁻⁵ kg.
The magnetic force must equal the weight, and the current is I = V/R:

$$BIl = mg \quad \Rightarrow \quad B\frac{V}{R}l = mg \quad \Rightarrow \quad V = \frac{mgR}{Bl}$$

$$V = \frac{6 \times 10^{-5} \times 9.8 \times 0.19}{0.6 \times 0.05} = \frac{1.12 \times 10^{-4}}{0.03} = \mathbf{3.7 \times 10^{-3}\ V}$$

about 3.7 mV (3.8 mV with g = 10 m/s²), which drives a current of about 20 mA.

$$\oint \vec{B}\cdot d\vec{l} = \int_a^b + \int_b^c + \int_c^d + \int_d^a$$

- Along **ab**, inside, the field is uniform and along the path: $\int_a^b = Bl$.
- Along **bc** and **da**, the path is at right angles to the field inside, and outside the field is zero: these give **0**.
- Along **cd**, outside, the field is (almost) zero: **0**.

So $\oint \vec{B}\cdot d\vec{l} = Bl$. The path encloses N = nl turns, each carrying I, so the current through it is NI = nlI. By Ampère's law,

$$Bl = \mu_0 nlI \qquad \Rightarrow \qquad \boxed{B = \mu_0 nI}$$

iii. A toroid

A **toroid** is a solenoid bent round into a ring, like a doughnut wound with wire: an endless solenoid. Its lines of force are circles **inside** the windings, centered on the toroid's center; there are no ends for the field to leak out of.

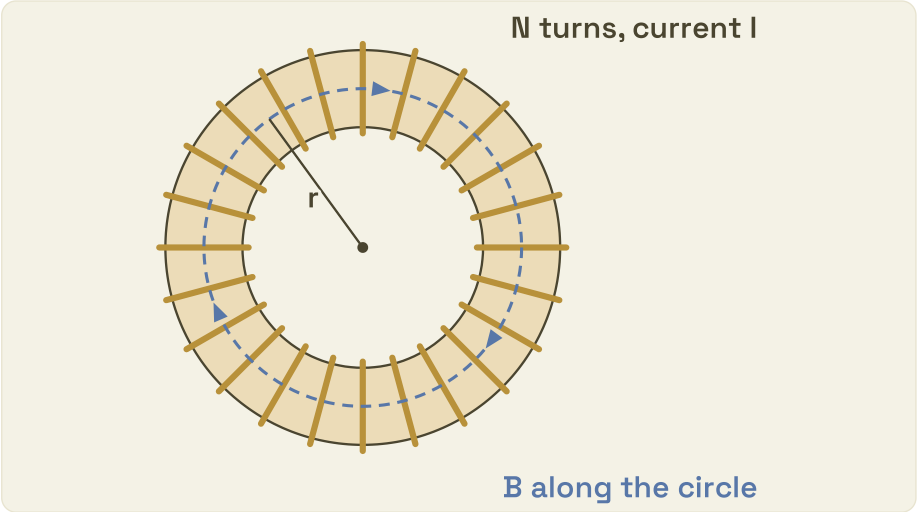


Figure 16.30 *A toroid of N turns. The dashed circle of radius r, inside the windings, is the path for Ampère's law: it is threaded by all N turns, each carrying I. The field inside the windings is along this circle.*

Take a toroid of N turns carrying a current I, and choose as the path a **circle of radius r** inside the windings, centered on the toroid's center. By symmetry, B is the same size all round it and along it:

$$\oint \vec{B}\cdot d\vec{l} = B \times 2\pi r$$

The circle is threaded by all **N turns**, each carrying I, so the current enclosed is NI:

$$B \times 2\pi r = \mu_0 NI \qquad \Rightarrow \qquad \boxed{B = \frac{\mu_0 NI}{2\pi r}}$$

Since $N/2\pi r = n$, the number of turns per unit length round the ring, $B = \mu_0 nI$: the same as a solenoid's, as it should be.

Two more results come from the same law. **Outside** the toroid, a circle round it encloses each turn twice, once going up and once coming down, so the net current is zero and **B = 0**. In the **hole in the middle**, a circle encloses no current at all: **B = 0** there too. All of a toroid's field is kept inside its windings, which is why toroidal transformers and chokes hum less and disturb nearby circuits less.

CAREFUL

How uniform is a toroid's field?

Inside the windings, B = μ_0NI/2πr depends on r: it is a little stronger near the inner edge of the ring than near the outer edge. For a **thin** toroid, whose ring is much narrower than its radius, r hardly changes across the windings, and the field is very nearly uniform. Along any one circle inside, its size is exactly the same all round.

There is. **Ampère's circuital law** is an alternative to the Biot–Savart law, and it is far quicker in cases with enough **symmetry**. It states:

The line integral of the magnetic field $\vec{B}$ round **any closed path** equals μ_0 times the **net current** passing through the area bounded by the path:

$$\boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enclosed}}}$$

Here the path is split into small steps $d\vec{l}$, and the products $B\,dl\cos\theta$ (the field's part along each step, times the step) are added all the way round. The result holds whatever the **size and shape** of the path, and it holds for **steady** currents. (Maxwell later added a term for changing electric fields, which led him to light as an electromagnetic wave.)

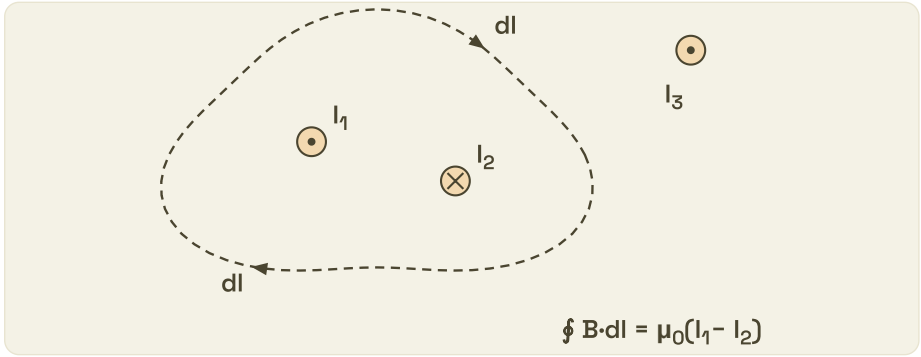


Figure 16.28 *Ampère's circuital law. Round any closed path (dashed), the sum of B·dl is μ_0 times the net current through it: here $I_1 - I_2$, since I_3 flows the other way. The current I_3 outside the path adds nothing to the sum, though its field is there along the path.*

The skill in using it is to choose a path along which B is the **same size** everywhere and either **along** the path or **at right angles** to it. Then the integral is just B times a length.

i. A long straight conductor

The lines of force round a long straight wire are circles centered on it. Choose as the path a **circle of radius r** round the wire. By symmetry, B has the same size everywhere on it and points along it, so

$$\oint \vec{B} \cdot d\vec{l} = B \times 2\pi r = \mu_0 I \qquad \Rightarrow \qquad B = \frac{\mu_0 I}{2\pi r}$$

the same result as the Biot–Savart law gave, in two lines.

ii. A long solenoid

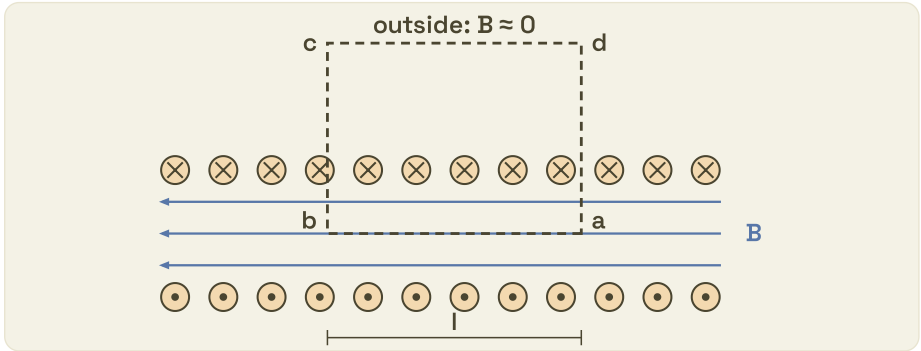


Figure 16.29 *A long solenoid cut along its axis, the current going into the page in the top layer of turns (⊗) and out of it in the bottom layer (⊙). The rectangular path abcd has its side ab, of length l, inside, along the axis; cd lies outside, where the field is about zero. It encloses N turns.*

Take a long solenoid with n turns per unit length carrying a current I. Choose a **rectangular path abcd**, with ab (of length l) inside the solenoid, parallel to the axis, and cd outside.

WORKED EXAMPLE

Example 16.12: a rod on a slope

A horizontal rod of mass 10 g and length 0.10 m lies on a smooth plane inclined at 60° to the horizontal. A uniform **vertical** magnetic field B acts in the region. Find B if the rod stays at rest on the plane when a current of 1.73 A flows in it.

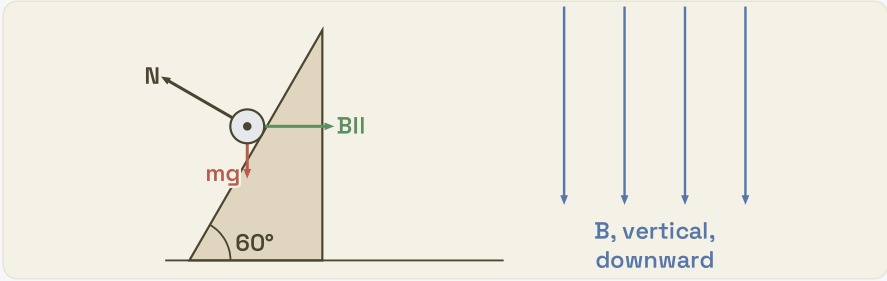


Figure 16.14 *The rod seen end-on, on a smooth slope at 60°. The vertical field and the current (out of the page) give a horizontal magnetic force BIl. Along the slope, its component BIl cos 60° (up the slope) balances the weight's component mg sin 60° (down the slope).*

Answer

The rod is horizontal and the field vertical, so θ = 90° and the magnetic force BIl is **horizontal** (at right angles to both), pointing into the slope. The plane is smooth, so only the components **along the slope** matter (the normal reaction takes care of the rest):

$$BIl\cos 60^\circ = mg\sin 60^\circ \quad \Rightarrow \quad B = \frac{mg\tan 60^\circ}{Il}$$

$$B = \frac{0.010 \times 9.8 \times 1.732}{1.73 \times 0.10} = \frac{0.170}{0.173} = \mathbf{0.98\ T}$$

about 1 T (exactly 1.0 T with g = 10 m/s²).

YOUR TURN

Test yourself 16.8: balancing a rod

What current is needed to balance a rod of mass 40 g and length 0.5 m placed across a magnetic field of 0.2 T? (g = 10 m/s²)

Answer

$$BIl = mg \quad \Rightarrow \quad I = \frac{mg}{Bl} = \frac{0.04 \times 10}{0.2 \times 0.5} = \mathbf{4\ A}$$

16.6 Force and torque on a rectangular coil in a uniform magnetic field

Why does a coil in a magnetic field turn, when the forces on it add up to zero?

When a current passes round a rectangular coil in a uniform magnetic field, the forces on its sides produce a **torque** that turns the coil until its plane is at right angles to the field. This turning is how every electric motor and every moving-coil meter works.

Take a rectangular coil PQRS of N turns, carrying a current I, in a uniform field B. Let its sides PQ and RS, of length l, be at right angles to the field, and its sides QR and SP have length b (the breadth). The coil can turn about an axis through its middle, parallel to PQ. Let the **plane of the coil** make an angle α with the field; then the **normal** to the coil makes an angle θ = 90° − α with it.

Step 1. The forces. Sides PQ and RS are at right angles to the field, so each feels a force $F = BIl$ (16.11) per turn. The currents in them are opposite, so the forces are **equal and opposite**, at right angles to the field.
Step 2. The other two sides. The forces on QR and SP are also equal and opposite, but they act **along the same line** (the axis of turning), so they cancel completely and give no torque.
Step 3. The torque. The forces on PQ and RS act along parallel lines a perpendicular distance $b\sin\theta$ apart, so they form a **couple**:

$$\tau = F \times b\sin\theta \quad \ldots (16.10)$$

$$\tau = BIl \times b\sin\theta = BIA\sin\theta$$

where $A = lb$ is the coil's area. For N turns,

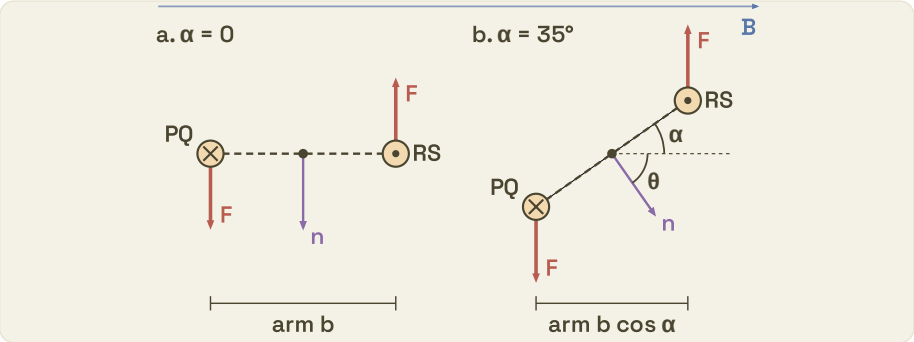


Figure 16.15 *A rectangular coil PQRS seen end on, along its axis of turning (the dot), in a uniform field B. Side PQ carries the current into the page (⊗) and is pushed down; RS carries it out of the page (⊙) and is pushed up: equal and opposite forces F along different lines, a couple. a. The coil's plane along the field (α = 0): the forces' lines are the whole breadth b apart. b. The plane at α to the field, its normal n at θ = 90° − α: the lines are b cos α = b sin θ apart.*

Step 4. Vector form. Define the coil's **magnetic dipole moment** $\vec{M} = NI\vec{A}$, of size NIA, along the normal to the coil (the way the right-hand grip rule gives from the current). Then

$$\vec{\tau} = \vec{M} \times \vec{B}, \qquad \tau = MB\sin\theta$$

The unit of magnetic moment is the **ampere meter squared** (A m²). This is the same law as for a bar magnet in a field (Unit 17): a current loop acts as a magnet whose moment is NIA.

Special cases:

i. The coil's **plane along the field** (α = 0°, θ = 90°): $\tau = NBIA$, the **largest** torque.

ii. The coil's **plane at right angles to the field** (α = 90°, θ = 0°): $\tau = 0$. The coil has turned as far as the field can turn it: its moment points along the field.

The energy of a coil, or a magnet, in a field

Turning a magnetic moment M against the field takes work. Turning it from angle θ_1 to θ_2 with the field, the torque MB sin θ does

$$W = \int_{\theta_1}^{\theta_2} MB\sin\theta\, d\theta = MB(\cos\theta_1 - \cos\theta_2)$$

so its **potential energy** is $U = -MB\cos\theta$: least when the moment is along the field (θ = 0, the **stable** position), greatest when it points against it (θ = 180°, an unstable balance). Turning it right round, from 0° to 180°, takes $W = 2MB$.

WORKED EXAMPLE

Example 16.13: largest and smallest torque

At what condition is the torque on a rectangular coil in a magnetic field greatest, and least?

Answer

For N turns of area A carrying a current I in a uniform field B, $\tau = NBIA\cos\alpha$, where α is the angle between the **plane** of the coil and the field.

- The torque is **greatest**, $\tau_{max} = NBIA$, when cos α = 1: α = 0°, the coil's plane **along** the field.
- It is **least**, $\tau_{min} = 0$, when cos α = 0: α = 90°, the coil's plane **at right angles** to the field.

WORKED EXAMPLE

Example 16.14: turning with no net force

Why does a current-carrying rectangular coil rotate in a uniform magnetic field, though the net force on it is zero?

Answer

The Lorentz forces on opposite sides of the coil are equal and opposite, so the **net force** is zero, and the coil as a whole doesn't move off. But the two forces on the sides PQ and RS don't act along the same line: they form a **couple**, whose torque $NBIA\cos\alpha$ turns the coil. (A zero net force means no linear acceleration; it doesn't mean no turning.)

WORKED EXAMPLE

Example 16.25: a square coil

Calculate the magnetic field at the center of a square coil of side 4 cm carrying a current of 5 A.

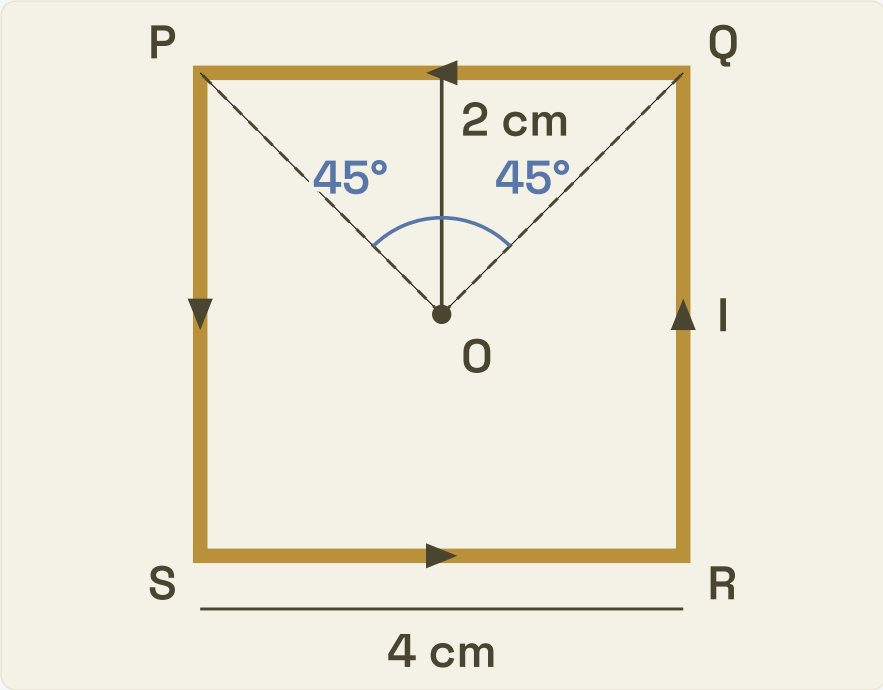


Figure 16.27 *A square coil PQRS of side 4 cm. The center O is 2 cm from each side, and the ends of each side are seen from O at 45° on either side of the perpendicular.*

Answer

Each side is a straight conductor, and the center O is at a perpendicular distance a = 2 cm = 0.02 m from it, with $\varphi_1 = \varphi_2 = 45°$. The field of side PQ at O:

$$B_1 = \frac{\mu_0 I}{4\pi a}(\sin 45^\circ + \sin 45^\circ) = \frac{10^{-7} \times 5}{0.02} \times \sqrt{2} = 3.54 \times 10^{-5}\ \text{T}$$

All four sides give fields at O in the **same direction** (at right angles to the coil, by the thumb rule), so

$$B = 4B_1 = 4 \times 3.54 \times 10^{-5} = \mathbf{1.41 \times 10^{-4}\ T}$$

(A circle of the same perimeter, 16 cm, would have r = 2.55 cm and B = $\mu_0 I/2r$ = 1.23 × 10⁻⁴ T, a little less.)

YOUR TURN

Test yourself 16.23: a power line

Calculate the magnetic field 5 m from a very long straight conductor carrying 100 A.

Answer

$$B = \frac{\mu_0 I}{2\pi a} = \frac{2 \times 10^{-7} \times 100}{5} = \mathbf{4 \times 10^{-6}\ T}$$

4 μT, about a twelfth of the Earth's field. (A power line's current alternates, so its field does too.)

16.10 Ampère's circuital law

Is there a quicker way to find the field when a current is arranged very symmetrically?

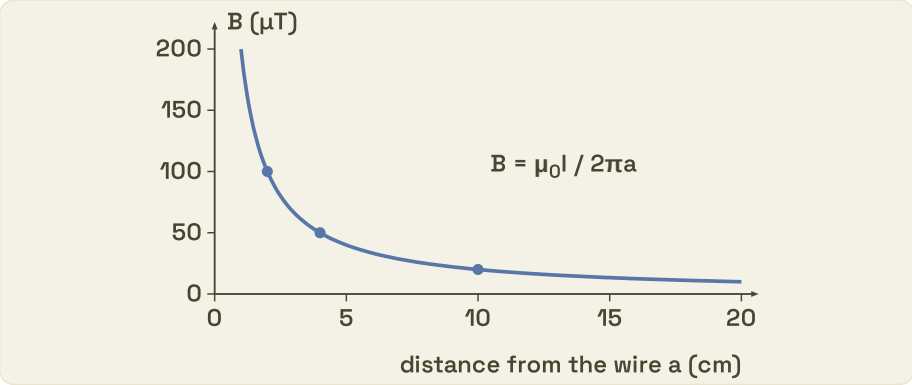


Figure 16.26 *The field of a long straight wire carrying 10 A, against the distance from it, from $B = \mu_0 I/2\pi a$: 100 μT at 2 cm, 50 μT at 4 cm, 20 μT at 10 cm.*

YOUR TURN

Test yourself 16.12: a bar on sloping rails

A uniform metal bar of mass 10 g rests across two parallel metal rails 10 cm apart, and a battery drives a current of 2 A through it. A magnetic field of 0.2 T acts at right angles to the plane of the rails. Through what angle must the rails be tilted so that the bar doesn't slide? (Ignore friction.)

Answer

The field is at right angles to the rails' plane and the bar lies across it, so the magnetic force BIl acts **along the rails**; with the current the right way, it acts up the slope. The bar stays put when it balances the weight's component down the slope:

$$BIl = mg\sin\theta \quad \Rightarrow \quad \sin\theta = \frac{BIl}{mg} = \frac{0.2 \times 2 \times 0.10}{0.010 \times 9.8} = \frac{0.04}{0.098} = 0.41$$

$$\theta = \mathbf{24^\circ}$$

YOUR TURN

Test yourself 16.13: torque on a square coil

A square coil of side 10 cm has 100 turns and carries a current of 2 A. It hangs in a magnetic field of 0.1 T. Find the torque on it when its plane i. is parallel to the field, ii. makes an angle of 60° with the field.

Answer

A = 0.10 × 0.10 = 0.01 m², N = 100, I = 2 A, B = 0.1 T. With α the angle between the plane and the field, τ = NBIA cos α.

i. $\alpha = 0^\circ$: $\tau = 100 \times 0.1 \times 2 \times 0.01 \times 1 = \mathbf{0.2\ N\,m}$

ii. $\alpha = 60^\circ$: $\tau = 0.2 \times \cos 60^\circ = \mathbf{0.1\ N\,m}$

16.7 Moving-coil galvanometer

A **galvanometer** is an instrument for detecting and measuring very small currents. The ammeters and voltmeters with a needle in a school lab are galvanometers inside, with resistors added (Grade 11, Unit 23).

Principle. A current-carrying coil in a magnetic field feels a torque (Section 16.6). In the galvanometer that torque is balanced by a spring, and the angle the coil turns through measures the current.

Construction. A rectangular coil of N turns of fine insulated copper wire, of length l and breadth b, is wound on a light non-magnetic frame (brass or aluminium). In the **suspended-coil** type, used for the most sensitive work, it hangs by a fine **phosphor-bronze strip** from a **torsion head** at the top, which can be turned to set the zero, and its lower end is joined to a loose **hairspring** of phosphor bronze. The current goes in and out through the suspension and the spring. The coil hangs between the concave **pole pieces** of a strong permanent magnet, round a fixed **soft-iron cylinder**, the **core**, so that the field in the gap is **radial**: it points along the radii of the core, towards its axis or away from it. A light mirror on the suspension (or a pointer) shows the deflection on a scale. In the **pivoted** type, used in lab meters, the coil turns on jewel bearings between two hairsprings and carries a pointer.

Working. With a current I in the coil, the forces on its two vertical sides form a couple. In a **radial** field, the field is always along the plane of the coil, whatever its deflection (α = 0 always), so the **deflecting torque** is always the largest value:

$$\tau = NBIA \quad \text{... (16.12)}$$

As the coil turns through an angle θ, the suspension (or the hairspring) is twisted and gives a **restoring torque** in proportion to the twist:

$$\tau_{\text{restoring}} = k\theta \quad \text{... (16.13)}$$

where k is the **torsion constant** of the suspension or spring (the torque per radian of twist). The coil comes to rest where the two torques balance:

$$NBIA = k\theta \qquad \Rightarrow \qquad I = \frac{k}{NBA}\theta = G\theta \quad \text{... (16.14)}$$

where $G = \dfrac{k}{NBA}$ is the **galvanometer constant**. So **I ∝ θ**: the deflection is in direct proportion to the current, and the scale can be marked evenly. This is the theory of the moving-coil galvanometer. Without the radial field, the torque would be NBIA cos α and fall off as the coil turned, and the scale would be crowded at one end.

Current sensitivity

The **current sensitivity** of a galvanometer is the deflection per unit current:

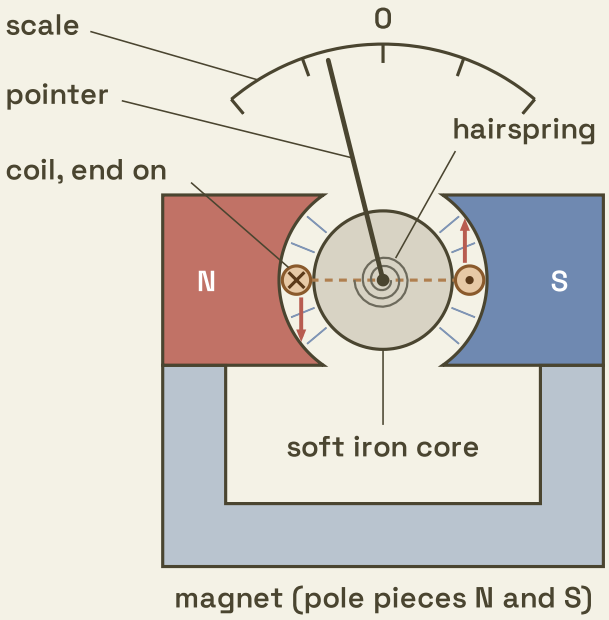


Figure 16.16 *A moving-coil galvanometer seen from the front. The coil's two sides sit in the gaps between the curved poles and the soft-iron core. Current goes into the page in the left side (⊗) and out in the right side (⊙): the left side is pushed down and the right side up, so the coil and its pointer turn until the hairspring holds them.*

A galvanometer is **sensitive** if a small current gives a large deflection. So it is more sensitive if:

i. the number of turns N is large;
ii. the magnetic field B is strong;
iii. the area of the coil A is large;
iv. the torsion constant k is small (a fine, soft suspension).

Voltage sensitivity

The **voltage sensitivity** is the deflection per unit potential difference across the galvanometer. If its coil has resistance R, then V = IR, and

$$\frac{\theta}{V} = \frac{\theta}{IR} = \frac{\theta}{I} \times \frac{1}{R} = \frac{NBA}{kR} \quad \dots (16.16)$$

It is higher for large N, B and A, small k, and **small resistance R**.

CAREFUL

More turns don't always help

Adding turns raises the current sensitivity, NBA/k. But the extra wire raises the coil's resistance R in proportion to N (the same thickness of wire), so the voltage sensitivity, NBA/(kR), stays about the same. Doubling the turns doubles the current sensitivity but leaves the voltage sensitivity unchanged.

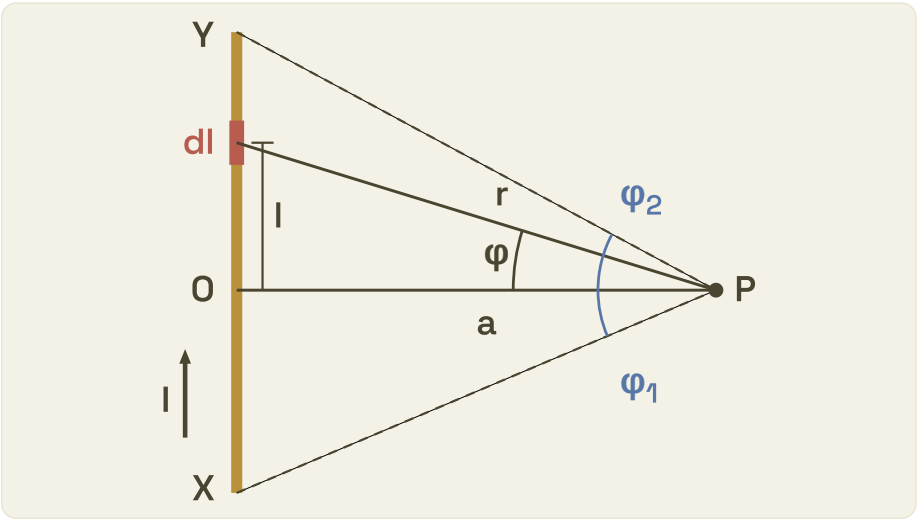


Figure 16.25 *The field of a straight conductor XY at a point P, a perpendicular distance a from it. An element dl at a distance l from the foot O is r from P and makes an angle φ with the perpendicular PO. The wire's two ends are seen from P at angles φ_1 and φ_2 on either side of PO.*

Step 1. An element dl at a distance l from O is a distance r from P, where $r^2 = a^2 + l^2$. Let the line from the element to P make an angle φ with PO; then $l = a\tan\varphi$ and $r = a\sec\varphi$, and the angle θ between the element and that line has $\sin\theta = \cos\varphi$.
Step 2. By the Biot–Savart law,

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl\cos\varphi}{r^2}$$

Every element's field at P points the same way (into the page, for the current in the figure), so they add.
Step 3. As l changes, $dl = a\sec^2\varphi\,d\varphi$, and

$$dB = \frac{\mu_0 I}{4\pi}\frac{a\sec^2\varphi\,\cos\varphi\,d\varphi}{a^2\sec^2\varphi} = \frac{\mu_0 I}{4\pi a}\cos\varphi\,d\varphi$$

Step 4. Adding (integrating) from the angle φ_1 on one side of PO to φ_2 on the other:

$$B = \frac{\mu_0 I}{4\pi a}\int_{-\varphi_1}^{\varphi_2}\cos\varphi\,d\varphi = \boxed{\frac{\mu_0 I}{4\pi a}(\sin\varphi_1 + \sin\varphi_2)}$$

Step 5. For an **infinitely long** wire, $\varphi_1 = \varphi_2 = 90°$, and

$$\boxed{B = \frac{\mu_0 I}{2\pi a}}$$

The field round a long straight wire is in inverse proportion to the distance from it. (For a wire that ends at the foot O and runs to infinity on one side, $\varphi_1 = 0$ and $\varphi_2 = 90°$, and B is half this, $\mu_0 I/4\pi a$.)

Graph. B against the distance from the wire is a falling curve, $B \propto 1/a$: strongest close to the wire, weaker farther away, halving each time the distance doubles.

Direction: circles round the wire, by the right-hand thumb rule.

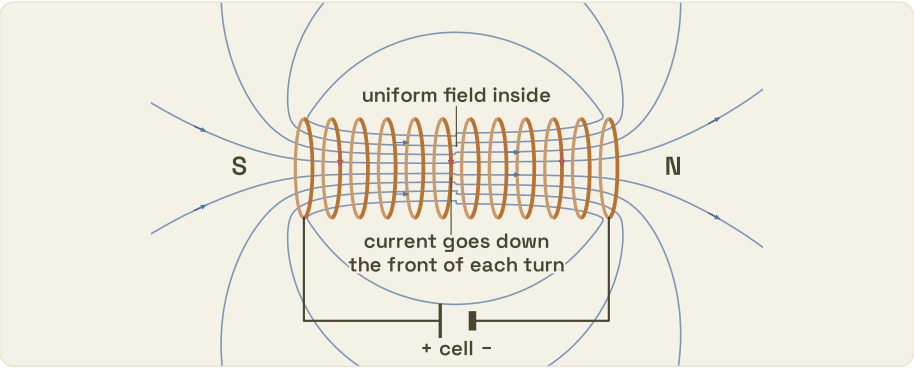


Figure 16.23 *The field of a solenoid. Inside, the lines are straight, parallel and evenly spaced: the field is uniform. Outside, the pattern is like a bar magnet's.*

Graph. Along the axis, the field is strong and uniform over most of the solenoid's length, falls to half near each end, and dies away outside.

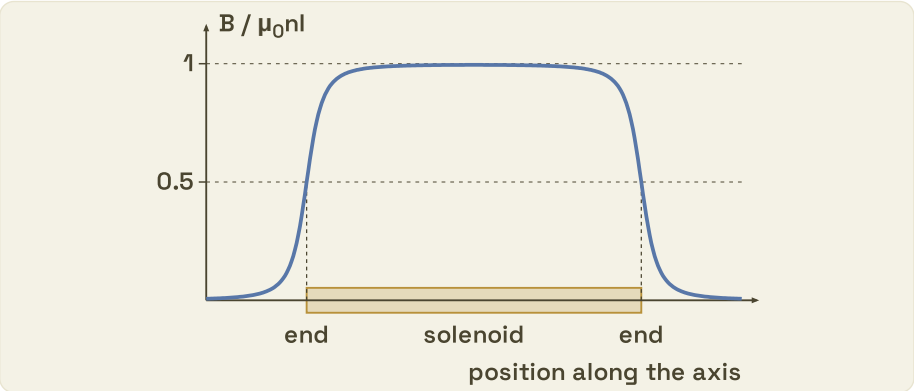


Figure 16.24 *The field on the axis of a solenoid 10 times as long as it is wide, worked out by adding up the fields of its turns, as a fraction of $\mu_0 nI$. It is uniform over most of the length, half at each end, and falls away outside.*

Direction: along the axis, by the right-hand grip rule.

WORKED EXAMPLE

Example 16.24: a wire inside a solenoid

A horizontal wire 5 cm long carrying a current of 2 A is placed in the middle of a long solenoid, at right angles to its axis. The solenoid has 1000 turns per meter and carries a steady current I. Calculate I if the force on the wire is vertically downward and equal to 10^{-4} N.

Answer

l = 5 cm = 0.05 m, current in the wire $I_w = 2$ A, n = 1000 per m, F = 10^{-4} N.
The solenoid's field at the wire is $B = \mu_0 n I$, at right angles to the wire, so the force on the wire is

$$F = BI_w l = \mu_0 n I\, I_w l \quad \Rightarrow \quad I = \frac{F}{\mu_0 n I_w l}$$

$$I = \frac{10^{-4}}{4\pi \times 10^{-7} \times 1000 \times 2 \times 0.05} = \frac{10^{-4}}{1.26 \times 10^{-4}} = \mathbf{0.80\ A}$$

iv. The field of a straight current-carrying conductor

Take a straight conductor carrying a current I, and a point P at a perpendicular distance a from it; let O be the foot of the perpendicular.

YOUR TURN

A galvanometer's current

A galvanometer coil has 50 turns, each 2 cm × 2 cm, in a radial field of 0.2 T. Its springs have a torsion constant of 2.5×10^{-6} N m per radian. What current turns it through 60°? What is its current sensitivity?

Answer

A = $0.02 \times 0.02 = 4 \times 10^{-4}$ m², so NBA = $50 \times 0.2 \times 4 \times 10^{-4} = 4 \times 10^{-3}$ N m/A. θ = 60° = π/3 = 1.047 rad.

$$I = \frac{k\theta}{NBA} = \frac{2.5 \times 10^{-6} \times 1.047}{4 \times 10^{-3}} = \mathbf{6.5 \times 10^{-4}\ A}\ (0.65\ \text{mA})$$

Current sensitivity: $\frac{NBA}{k} = \frac{4 \times 10^{-3}}{2.5 \times 10^{-6}} = \mathbf{1600\ rad/A}$, or 1.6 rad (about 92°) per milliampere.

16.8 Hall effect

If the field pushes the moving electrons in a wire sideways, shouldn't one side of the wire become charged?

It does, very slightly. In 1879 **Edwin Hall**, a student at Johns Hopkins University in the United States, passed a current along a thin strip of gold leaf in a strong magnetic field and found a small voltage **across** the strip, at right angles to the current. When a magnetic field is applied at right angles to a current-carrying conductor, a voltage is developed across the conductor in the direction at right angles to both the current and the field. This is the **Hall effect**, and the voltage is the **Hall voltage**, V_H.

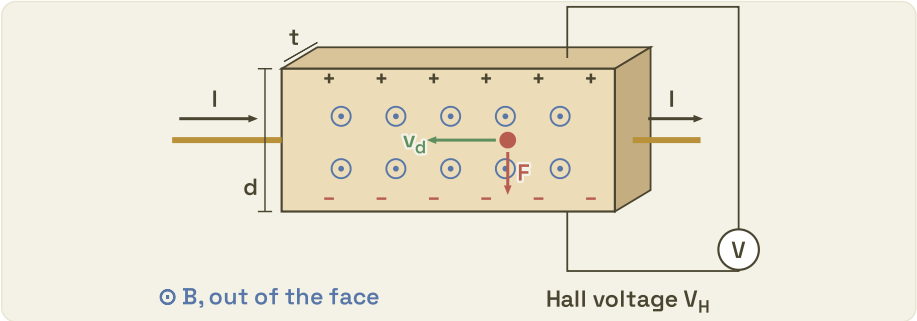


Figure 16.17 *The Hall effect in a slab of width d and thickness t. The current I flows to the right (+x), so the free electrons drift to the left with speed v_d. The field B is along +z, out of the slab's broad face, towards you. Each electron is pushed down (−y) by the force Bev_d, so electrons pile up on the lower face and leave the upper face positive: the Hall voltage V_H appears across the width d.*

Take a slab of conductor of width d and thickness t, so its cross-section is A = dt. A current I flows through it from left to right, along the +x-axis, so its free electrons drift from right to left with speed v_d. A uniform field B is applied along the z-axis, at right angles to the slab's broad face.

Step 1. Each electron feels the Lorentz force $F_m = Bev_d$, which (for a negative charge, Fleming's rule reversed) is vertically **downward**. Electrons collect on the **lower face**, making it negative and leaving the **upper face** positive.

Step 2. The charges on the two faces set up an electric field E across the slab, pointing down, and a potential difference V_H between the faces:

$$E = \frac{V_H}{d}$$

This field pushes each electron **upward** with a force $F_e = eE = \frac{eV_H}{d}$.

Step 3. Electrons go on collecting until the two forces balance, which happens almost at once. Then

$$F_e = F_m \quad \Rightarrow \quad \frac{eV_H}{d} = Bev_d \quad \Rightarrow \quad V_H = Bdv_d \quad \ldots (16.17)$$

Step 4. The current is $I = nev_d A = nev_d\, dt$, where n is the number of free electrons per unit volume, so

$$v_d = \frac{I}{ne\, dt} \quad \ldots (16.18)$$

Step 5. Putting (16.18) into (16.17):

$$V_H = Bd\,\frac{I}{ne\, dt} \quad \Rightarrow \quad \boxed{V_H = \frac{BI}{net}}$$

where t is the thickness of the slab **along the field**. The Hall voltage is larger for a **thinner** slab and for a material with **fewer** charge carriers per unit volume (smaller n), and it is in direct proportion to B.

DID YOU KNOW?

Small n, large voltage

In a metal such as copper, n is about 10^{29} per cubic meter; in a semiconductor such as indium antimonide it is about 10^{22} per cubic meter, ten million times fewer. With the same current, the carriers in the semiconductor must drift ten million times faster, and the Hall voltage is ten million times larger. That is why Hall sensors are made of semiconductors.

The **sign** of the Hall voltage tells the sign of the carriers. In an n-type semiconductor the carriers are electrons and the upper face in the figure is positive; in a p-type one the carriers are holes, which drift the other way and are pushed to the same (lower) face, so that face becomes positive instead. The Hall effect was how physicists first learned that some materials carry current by positive carriers.

Measuring a magnetic field with a Hall probe

A **Hall probe** measures magnetic flux density by the Hall effect. It is a thin slice of semiconductor at the flat end of a rod, with a steady current through it and leads to a voltmeter across it. Rearranging $V_H = BI/(net)$,

$$B = \frac{V_H\, net}{I}$$

For a given probe and current, n, e, t and I are fixed, so **B is in direct proportion to** V_H, and the meter can be marked in teslas. For an accurate reading, the flat face must be held **at right angles** to the field lines: the reading is largest then, smaller when the probe is tilted, and zero when its face is along the field. A good Hall probe can detect even the Earth's field.

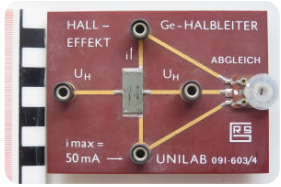


Figure 16.18 *A school Hall-effect board (its labels in German). The gray slip in the middle is a thin wafer of n-type germanium: the current i goes along it between the top and bottom sockets, and the Hall voltage, U_H, is taken across it from the left and right sockets. The knob trims the reading to zero when there is no field. The scale at the left is in centimeters.*

Photo: Panoga, CC BY-SA 4.0

WORKED EXAMPLE

Example 16.15: carriers in indium antimonide

A slice of indium antimonide is 2.5 mm thick and carries a current of 150 mA. A magnetic field of flux density 0.5 T, correctly applied, produces a maximum Hall voltage of 8.75 mV between the edges of the slice. Calculate the number of free charge carriers per unit volume, assuming each has a charge of -1.6×10^{-19} C.

Answer

t = 2.5 mm = 2.5×10^{-3} m, I = 150 mA = 0.150 A, B = 0.5 T, V_H = 8.75 mV = 8.75×10^{-3} V, e = 1.6×10^{-19} C.

$$V_H = \frac{BI}{net} \quad \Rightarrow \quad n = \frac{BI}{V_H e t} = \frac{0.5 \times 0.150}{8.75 \times 10^{-3} \times 1.6 \times 10^{-19} \times 2.5 \times 10^{-3}} = \frac{0.075}{3.5 \times 10^{-24}}$$

$$n = \mathbf{2.14 \times 10^{22}\ per\ m^3}$$

WORKED EXAMPLE

Example 16.16: electrons in copper

A slab of copper 2 mm thick and 1.50 cm wide is placed in a uniform magnetic field of flux density 0.40 T, so that the most flux passes through it. When a current of 75 A flows through it, a potential difference of 0.81 μV is developed between its edges. Find the number of mobile electrons per unit volume in copper.

Answer

t = 2 mm = 2×10^{-3} m (the thickness along the field), B = 0.40 T, I = 75 A, V_H = 0.81 μV = 0.81×10^{-6} V. (The width, 1.50 cm, isn't needed.)

$$n = \frac{BI}{V_H e t} = \frac{0.40 \times 75}{0.81 \times 10^{-6} \times 1.6 \times 10^{-19} \times 2 \times 10^{-3}} = \frac{30}{2.59 \times 10^{-28}} = \mathbf{1.16 \times 10^{29}\ per\ m^3}$$

A large current and a strong field give less than a microvolt in copper: n is so large that the electrons drift very slowly.

at right angles to the line from the element to P.

Step 2. Split dB into two parts: $dB\sin\varphi$ **along** the axis and $dB\cos\varphi$ **at right angles** to it, where φ is the angle between the line from the element to P and the axis, so that $\sin\varphi = a/r$.

Step 3. The element diametrically opposite gives a field of the same size whose part at right angles to the axis points the other way. Taken in pairs round the coil, the **parts at right angles to the axis cancel**, and only the parts **along the axis** add up.

Step 4. Adding the parts along the axis, with $\sum dl = 2\pi a$:

$$B = \sum dB\sin\varphi = \frac{\mu_0 I}{4\pi r^2} \cdot \frac{a}{r} \cdot 2\pi a = \frac{\mu_0 I a^2}{2r^3}$$

For N turns,

$$\boxed{B = \frac{\mu_0 N I a^2}{2\left(a^2 + x^2\right)^{3/2}}}$$

Check: at the center, x = 0, and $B = \frac{\mu_0 N I a^2}{2a^3} = \frac{\mu_0 N I}{2a}$, as before.

Far away, where x ≫ a, $(a^2 + x^2)^{3/2} \approx x^3$, and

$$B \approx \frac{\mu_0 N I a^2}{2x^3}$$

Far from the coil the field falls as **1/x³**, just like the field on the axis of a small bar magnet: a current loop is a magnetic dipole.

The field is along the axis only **on the axis**. At points off the axis, the field is not zero: it has parts both along and across the axis, and the lines of force loop round each side of the wire, like a bar magnet's. (Working it out off the axis takes harder mathematics.)

Graph. B against x is **bell-shaped**, the same on both sides of the coil: largest at the center, falling off on both sides, never quite reaching zero.

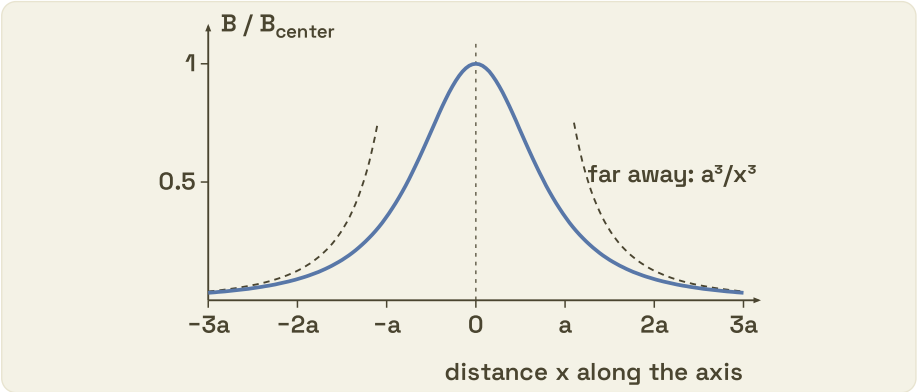


Figure 16.22 *The field on the axis of a circular coil, worked out from the formula, as a fraction of its value at the center, against the distance from the center in units of the radius a. At x = a it is 0.35 of the center's value; at x = 2a, 0.09. The dashed curve is the far-away formula, $\mu_0 NIa^2/2x^3$, which is close beyond about x = 2a.*

Direction: along the axis, by the right-hand grip rule: curl the fingers of your right hand round the coil along the current, and the thumb gives the field.

iii. The field of a long solenoid

A long coil of closely packed turns is a **solenoid**. Its field is like a bar magnet's: one end acts as a north pole and the other as a south pole. **Inside**, well away from the ends, the field is strong, **uniform** and **along the axis**. **Outside**, the field is very weak compared with that inside, and for a long solenoid it can be taken as zero.

Adding up the fields of all its turns by the Biot–Savart law gives, inside a long solenoid,

$$\boxed{B = \mu_0 n I}$$

where n is the number of turns **per unit length**, n = N/l. (Section 16.10 shows the same result more simply, by Ampère's law.) B is in direct proportion to n and to I, and does not depend on the solenoid's radius. At either **end** of a long solenoid, on the axis, the field is **half** that inside, $\frac{1}{2}\mu_0 nI$: the end of a long solenoid has turns on only one side of it. If the solenoid is wound on a core of relative permeability μ_r, the field is μ_r times larger, $\mu_0\mu_r nI$ (Unit 17).

WORKED EXAMPLE

Example 16.22: one loop or four

A wire 1 m long carries a steady current. Bent into one circular loop, it gives a field B at the loop's center. The same wire is then bent into a smaller coil of four turns, with the same current. Find the field at the center of the new coil.

Answer

One loop: $2\pi r = l$, so $r = \dfrac{l}{2\pi}$, and $B = \dfrac{\mu_0 I}{2r} = \dfrac{\pi\mu_0 I}{l}$ … (i)

Four turns: $4 \times 2\pi r' = l$, so $r' = \dfrac{l}{8\pi}$, and

$$B' = \frac{4\mu_0 I}{2r'} = \frac{4\mu_0 I \times 8\pi}{2l} = \frac{16\pi\mu_0 I}{l} \quad \text{… (ii)}$$

Dividing (ii) by (i): $\dfrac{B'}{B} = 16$, so $\mathbf{B' = 16\ B}$. Four times the turns, each a quarter of the radius: 4×4.

WORKED EXAMPLE

Example 16.23: two coils at right angles

Two identical coils carrying equal currents have a common center, and their planes are at right angles to each other. What is the ratio of the field at the center to the field of one coil alone?

Answer

Each coil alone gives $B_1 = B_2 = \dfrac{\mu_0 I}{2r}$ at the center, along its own axis. The two axes are at right angles, so the fields add as perpendicular vectors:

$$B = \sqrt{B_1^2 + B_2^2} = \sqrt{2B_1^2} = \sqrt{2}\,B_1$$

The ratio $B/B_1 = \sqrt{2}$, and the resultant is at 45° to each coil's axis.

YOUR TURN

Test yourself 16.20: the field of an orbiting electron

In an atom, the radius of the electron's orbit is 4×10^{-11} m, and the electron goes round 10^{15} times a second. Its charge is 1.6×10^{-19} C. Calculate the magnetic field at the center of the orbit.

Answer

$$B = \frac{\mu_0 ef}{2r} = \frac{4\pi \times 10^{-7} \times 1.6 \times 10^{-19} \times 10^{15}}{2 \times 4 \times 10^{-11}} = \frac{2.01 \times 10^{-10}}{8 \times 10^{-11}} = \mathbf{2.5\ T}$$

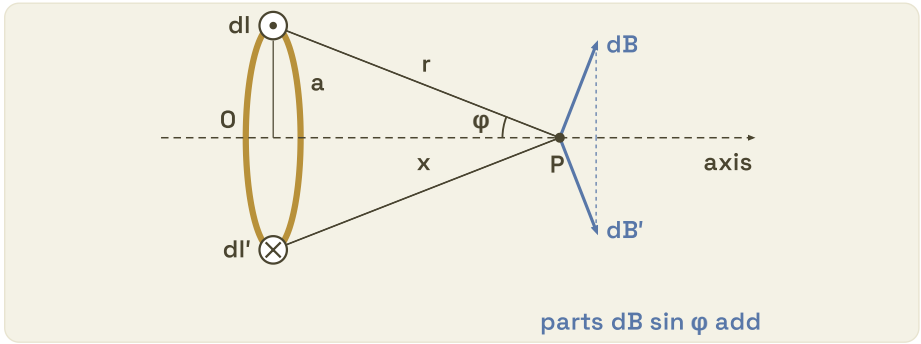


Figure 16.21 *The field at a point P on the axis of a circular coil. The element dl at the top is a distance $r = \sqrt{(a^2 + x^2)}$ from P, at right angles to the line to P; its field dB is at right angles to that line. The element at the bottom, dl′, gives dB′. Their parts at right angles to the axis (dB cos φ) cancel; their parts along the axis (dB sin φ) add.*

Step 1. Take a small element of length dl at the top of the coil, a distance $r = \sqrt{a^2 + x^2}$ from P. The line from it to P is at right angles to it ($\theta = 90°$), so

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl}{r^2}$$

YOUR TURN

Test yourself 16.18: a second field

The Hall voltage across a conductor in a 58 mT magnetic field is 2.0 μV. With the same current in a different field, the voltage is 2.5 μV. What is the strength of the second field?

Answer

With the same conductor and current, $V_H \propto B$:

$$B_2 = B_1\frac{V_2}{V_1} = 58 \times \frac{2.5}{2.0} = \mathbf{72.5\ mT}$$

YOUR TURN

Test yourself 16.19: a thin copper film

An electromagnet supplies a field of 1 T across a copper film carrying a current of 8 A. The film is 8×10^{-4} mm thick. Taking the number of free electrons per cubic meter of copper to be 10^{29}, find the Hall voltage.

Answer

$t = 8 \times 10^{-4}$ mm $= 8 \times 10^{-7}$ m.

$$V_H = \frac{BI}{net} = \frac{1 \times 8}{10^{29} \times 1.6 \times 10^{-19} \times 8 \times 10^{-7}} = \frac{8}{1.28 \times 10^{4}} = \mathbf{6.3 \times 10^{-4}\ V}$$

about 0.6 mV. Even in a film less than a thousandth of a millimeter thick, copper gives well under a millivolt.

16.9 Biot–Savart law

Oersted's needle showed that a current makes a field. How strong is that field, at a given place?

In 1820, a few months after Oersted's discovery, the French physicists **Jean-Baptiste Biot** and **Félix Savart** measured the field of a current, and **Pierre-Simon Laplace** put their results into a law for a small piece of the current. It is called the **Biot–Savart law** (or Laplace's law). With it, the field of any shape of wire can be worked out by adding up its pieces.

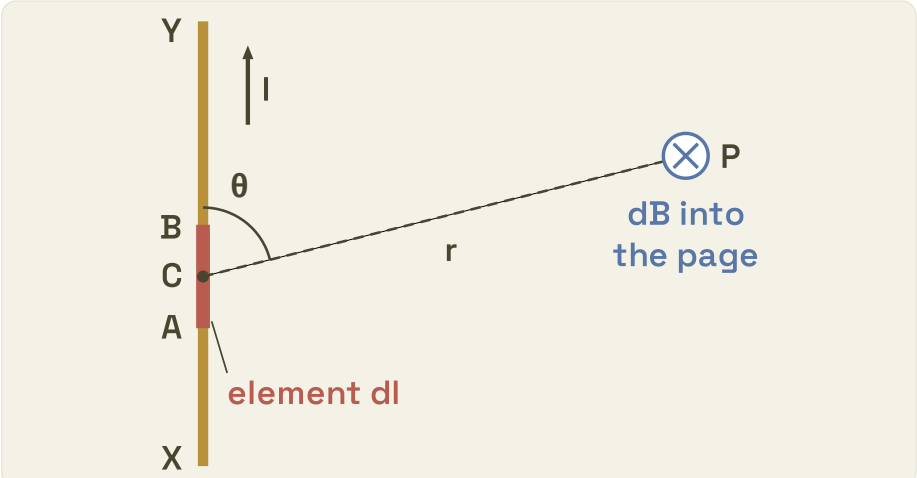


Figure 16.19 *The Biot–Savart law. A conductor XY carries a current I. A small element AB of it, of length dl, with its center at C, makes a field dB at a point P, a distance r from C. θ is the angle between the element and the line CP. By the right-hand thumb rule, dB at P points into the page (⊗).*

Take a conductor XY carrying a current I, a small element AB of it of length dl, and a point P at a distance r from the element's center C. Let θ be the angle between the element and the line CP. The field dB at P due to this element is:

i. directly proportional to the current: $dB \propto I$
ii. directly proportional to the element's length: $dB \propto dl$
iii. directly proportional to the sine of the angle θ: $dB \propto \sin\theta$

iv. inversely proportional to the square of the distance: $dB \propto \frac{1}{r^2}$

Combining these, $dB \propto \frac{I\,dl\sin\theta}{r^2}$, or $dB = k\frac{I\,dl\sin\theta}{r^2}$, where the constant k depends on the units. In SI, $k = \frac{\mu_0}{4\pi}$:

$$\boxed{dB = \frac{\mu_0}{4\pi}\frac{I\,dl\sin\theta}{r^2}}$$

Here μ_0 is the **permeability of free space**, $\mu_0 = 4\pi \times 10^{-7}$ H/m (henry per meter, the same as Wb/(A m) or N/A²); so $\mu_0/4\pi = 10^{-7}$ H/m. (In the old CGS electromagnetic units, k = 1.)

Direction of dB. It is at right angles to both the element and the line CP, given by the right-hand thumb rule. In the figure, it points into the page at P. In vector form, $d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{l}\times\hat{r}}{r^2}$.

Note two things. A point on the line of the element itself ($\theta = 0°$ or 180°) gets **no field** from it. And the law has the same inverse-square form as Coulomb's law for the electric field of a charge.

Check 8

A short element of a current-carrying wire makes a field dB at a point 2 cm away, at right angles to the element. At a point 4 cm away, at 30° to the element (on the other side), the field from the same element is:

a dB/2 b dB/4 c dB/8 d dB/16

Applications of the Biot–Savart law

i. The field at the center of a circular coil

Take a circular loop of radius r, center O, carrying a current I. Every element dl of the loop is at the same distance r from O, and the line from it to O is **at right angles** to it ($\theta = 90°$, $\sin\theta = 1$). So each element gives

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl}{r^2}$$

and, by the right-hand thumb rule, all the dB at O point the **same way**, along the coil's axis. Adding them up, $\sum dl = 2\pi r$, the loop's circumference:

$$B = \frac{\mu_0 I}{4\pi r^2} \times 2\pi r = \frac{\mu_0 I}{2r} \quad \text{(one turn)}$$

For a coil of N turns, close together,

$$\boxed{B = \frac{\mu_0 NI}{2r}}$$

Direction: along the coil's axis, by the right-hand grip rule. Seen from one side, a current going **anticlockwise** gives a field coming **out** towards you (that face is a north pole); clockwise gives a field going away from you (a south pole).

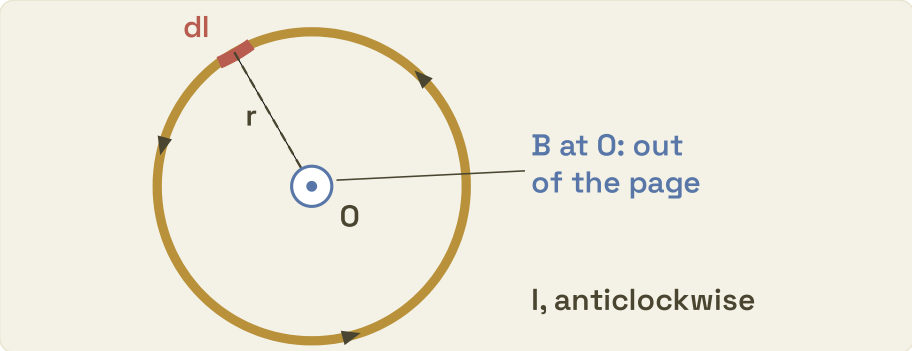


Figure 16.20 *The field at the center O of a circular coil of radius r. Every element dl is at the same distance r and at right angles to the line to O, and all their fields at O point the same way: out of the page for this anticlockwise current.*

For an arc that is only a fraction of a circle, the elements add up in the same way, but there are fewer of them: an arc that subtends an angle α (in radians) at its center gives $B = \frac{\mu_0 I}{2r}\cdot\frac{\alpha}{2\pi}$.

WORKED EXAMPLE

Example 16.17: a very large loop

Explain why the magnetic field at the center of a circular coil vanishes for an infinitely large loop.

Answer

The field at the center of a circular loop of radius a is $B = \frac{\mu_0 NI}{2a}$. As a → ∞, B → 0. Every element's contribution falls off as 1/a², while the number of elements grows only as a: the total falls as 1/a. Each part of an enormous loop is too far from the center to matter.

WORKED EXAMPLE

Example 16.18: a coil of 50 turns

A circular coil has 50 turns and a mean radius of 10 cm. If it carries a current of 3 A, find the magnetic field at its center.

Answer

N = 50, r = 10 cm = 0.10 m, I = 3 A, $\mu_0 = 4\pi \times 10^{-7}$ H/m.

$$B = \frac{\mu_0 NI}{2r} = \frac{4\pi \times 10^{-7} \times 50 \times 3}{2 \times 0.10} = \mathbf{9.4 \times 10^{-4}\ T}$$

WORKED EXAMPLE

Example 16.19: the electron in a hydrogen atom

In Bohr's model of the hydrogen atom, the electron goes round the nucleus in a circle of radius 5.1×10^{-11} m at 6.8×10^{15} revolutions per second. What magnetic field does it set up at the center of its orbit?

Answer

r = 5.1×10^{-11} m, f = 6.8×10^{15} rev/s, e = 1.6×10^{-19} C.

The electron passes any point of its orbit f times a second, carrying charge e each time, so it is a current $I = ef$.

$$B = \frac{\mu_0 I}{2r} = \frac{\mu_0 ef}{2r} = \frac{4\pi \times 10^{-7} \times 1.6 \times 10^{-19} \times 6.8 \times 10^{15}}{2 \times 5.1 \times 10^{-11}} = \frac{1.37 \times 10^{-9}}{1.02 \times 10^{-10}} = \mathbf{13.4\ T}$$

A field stronger than a hospital MRI scanner's, made by a single electron, because it is so close.

WORKED EXAMPLE

Example 16.20: an alpha particle

An alpha particle makes one full turn round a circle of radius 1 m in 2 s. Calculate the magnetic field at the center of the circle. ($\mu_0 = 4\pi \times 10^{-7}$ H/m)

Answer

f = 1/T = 1/2 = 0.5 rev/s, q = 2e = 3.2×10^{-19} C, r = 1 m. The current is I = qf.

$$B = \frac{\mu_0 qf}{2r} = \frac{4\pi \times 10^{-7} \times 3.2 \times 10^{-19} \times 0.5}{2 \times 1} = \mathbf{1.0 \times 10^{-25}\ T}$$

WORKED EXAMPLE

Example 16.21: winding a coil

A copper wire 28 m long is wound into a flat circular coil 8.0 cm in diameter. If a current of 4.50 A flows through it, what is the magnetic field at its center?

Answer

r = 8.0/2 = 4.0 cm = 0.040 m, I = 4.50 A.

The number of turns: $N = \frac{\text{length}}{2\pi r} = \frac{28}{2\pi \times 0.040} = 111$ turns.

$$B = \frac{\mu_0 NI}{2r} = \frac{4\pi \times 10^{-7} \times 111 \times 4.50}{2 \times 0.040} = \mathbf{7.8 \times 10^{-3}\ T}$$

ii. The field on the axis of a circular coil

Take a circular coil of radius a, center O, carrying a current I, and a point P on its axis at a distance x from O.