Unit 8

Waves in Pipes and Strings

This unit online, with its interactives:
science-playground.com/s/12f8

A Gandharva musician draws a bow across the strings of a sarangi and slides a fingernail along them; a bansuri player lifts one finger and the note jumps. Blow across the mouth of a test tube and it hoots; pour in some water and the hoot goes higher. Every one of these is a stationary wave (Unit 6) trapped between two ends, and every note it can play is fixed by where its nodes and antinodes are allowed to sit.

This unit fits stationary waves into **air columns**, the organ pipes, flutes and resonance tubes, and onto **stretched strings**, the sitar, sarangi and sonometer. It finds the notes each can play, explains why a flute sounds richer than a pipe closed at one end, corrects for the air that spills out of an open end, and arrives at the three laws that every string player uses without thinking.

A STORY FROM SCIENCE

The baron who explained the blue sky

John William Strutt, Lord Rayleigh (1842–1919), was an English physicist and the third Baron Rayleigh. He did much of his work in a laboratory in his own country house. His two-volume *The Theory of Sound* (1877–78) gathered what was known about vibrations, strings, pipes and rooms, and much of it is still how the subject is taught; in it he worked out, among much else, how far the vibrating air reaches beyond the open end of a pipe, the **end correction** of this unit.

He also explained why the sky is blue: air molecules scatter short (blue) wavelengths of sunlight far more than long (red) ones, now called **Rayleigh scattering**. Weighing nitrogen with great care, he noticed that nitrogen taken from the air was slightly heavier than nitrogen made from chemicals. With William Ramsay he tracked down the cause, a gas no one had known: **argon**. For it he received the Nobel Prize in Physics in 1904 (Ramsay received the chemistry prize the same year). He was President of the Royal Society from 1905 to 1908.

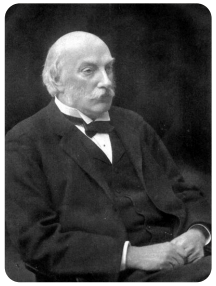


Figure 8.1 *Lord Rayleigh (1842–1919).*
Photo: Unknown author, Public domain

8.1 Introduction

Organ pipes are hollow tubes used as wind instruments to make sound. They can be made of wood, plastic or metal, and are mainly of three shapes: cylindrical, conical and rectangular. To sound a pipe, air is blown into it at one end, which sets the air column inside vibrating.

A longitudinal wave travels from that end along the pipe and is reflected at the other end. The wave going one way and the reflected wave coming back superpose and form a **stationary wave**. Organ pipes are designed to set up such a stationary wave in their air column. They **resonate** at particular frequencies, the natural frequencies of the air column, which depend on the pipe's length, its shape and how it is made. The air flowing in at the mouth keeps the column vibrating at one of these frequencies.

Organ pipes are of two types: **open** and **closed**.

8.2 Open and closed organ pipes

Open organ pipe

A pipe with **both ends open** is an **open organ pipe**. It has an antinode (the greatest displacement) at each open end. Its frequency depends on its length and its diameter. A flute, and the organ's octave pipes and open diapason, are open pipes.

Closed organ pipe

A pipe with **one end closed** and the other open is a **closed organ pipe**. It has an antinode at the open end and a node at the closed end. Its frequency, too, depends on its length and diameter. A resonance tube, an organ's stopped pipes (capped at the top, as in the stopped diapason) and a test tube you blow across are closed pipes. (A round bottle with a narrow neck is different: its air works as a spring and its neck's air as a mass, a *Helmholtz resonator*, so its note doesn't follow the pipe rules.)

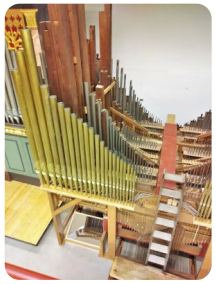


Figure 8.2 *Inside an organ. Each pipe plays one note: along each row the pipes get shorter and their notes higher. The tall square pipes at the back are made of wood.*

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DID YOU KNOW?

The bansuri is an open pipe

The bansuri, the bamboo flute heard all over Nepal, is plugged with a cork just above its mouth hole. You might expect it to act as a closed pipe, but the hole the player blows across opens the air column to the outside, and the far end is open too, so it behaves as a pipe **open at both ends**. Lifting a finger from a hole lets the air column "end" at that hole: the column is shorter, and the note higher.

8.3 Harmonics and overtones

The lowest frequency at which a vibrating system (a string, an air column) naturally vibrates is its **fundamental frequency**, f_1.

Harmonics are the whole-number multiples of the fundamental frequency. If the fundamental of a string or an air column is 100 Hz, its harmonics are: first harmonic 100 Hz (the fundamental itself), second harmonic 200 Hz, third harmonic 300 Hz, fourth harmonic 400 Hz, and so on.

Overtones are the frequencies, higher than the fundamental, that a system actually produces. The fundamental is not counted as an overtone. For a string or air column of fundamental 100 Hz that can produce every harmonic: first overtone 200 Hz (the second harmonic), second overtone 300 Hz (the third harmonic), third overtone 400 Hz (the fourth harmonic), and so on.

20.

a. Mention the factors affecting the velocity of transverse waves along a stretched string.

b. Explain the laws of vibration of a stretched string.

c. How do length, tension and mass per unit length affect the fundamental frequency of transverse vibration of a string?

d. A string of length 1.2 m has a mass of 0.01 kg and a tension of 100 N. Find the velocity of the transverse wave.

21.

a. A string is vibrating in its third harmonic. Sketch the wave pattern and explain the number of nodes and antinodes formed.

b. Find the frequency of the fundamental mode of a stretched string 1 m long, under a tension of 150 N, with mass per unit length 0.005 kg/m.

c. A stretched string vibrates in its second overtone at 300 Hz. What is its fundamental frequency?

A companion to your textbook, not a replacement for it: study from the book your school uses, and where the two differ, follow your teacher. Found a mistake? Please tell us.

	Harmonics	Overtones
Meaning	whole-number multiples of the fundamental frequency, nf_1	the frequencies a system produces above its fundamental
Use	mainly in physics, to name vibrational modes by their frequency	in acoustics and music, for the notes above the fundamental
Counting	1st (the fundamental), 2nd, 3rd, 4th…	1st overtone, 2nd overtone…; the fundamental isn't one
Example, fundamental 100 Hz, all harmonics present	1st 100 Hz, 2nd 200 Hz, 3rd 300 Hz	1st overtone 200 Hz (2nd harmonic), 2nd overtone 300 Hz (3rd harmonic)

CAREFUL

The overtones of a closed pipe

"The first overtone is the second harmonic" is true only for a system that makes every harmonic, a string or an open pipe. A closed pipe makes only the **odd** harmonics (section 8.4), so its first overtone is the **third** harmonic, $3f_1$, and its second overtone the fifth harmonic, $5f_1$. In an exam, find which harmonics the system makes first, then count the overtones among them.

8.4 Stationary waves in closed organ pipe

Hold a vibrating tuning fork just above the open end of a closed pipe. The air column starts vibrating at that end; the vibration travels down to the closed end and is reflected. The wave going down and the wave coming back superpose and form a **stationary longitudinal wave**. The closed end is always a node (the air there can't move) and the open end always an antinode, but between them the column can hold one, two, three or more quarter-wavelengths, so even a pipe of one length can vibrate in different **modes**, each with its own frequency. The modes differ in their number of nodes and antinodes. Let the air column be l long and let v be the speed of sound in the air (Figure 8.3).

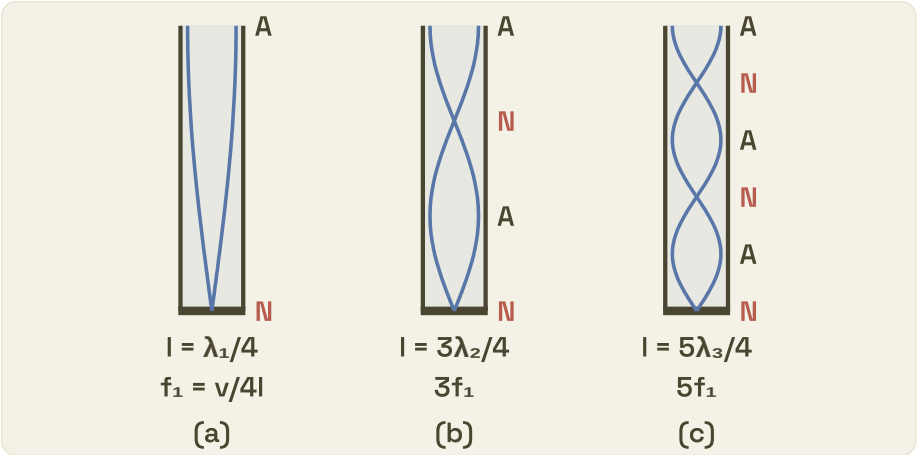


Figure 8.3 *The first three modes of a pipe closed at the bottom, drawn upright. The curves show how far the air moves at each height (it moves along the pipe; the curves are drawn sideways). The closed end is always a node, N, and the open end an antinode, A.*

First, or fundamental, mode. The column vibrates at its lowest possible frequency, with the longest possible wavelength: a node at the closed end, an antinode at the open end, and nothing in between (Figure 8.3 a). If λ_1 is the wavelength and f_1 the frequency,

$$\frac{\lambda_1}{4} = l \quad \Rightarrow \quad \lambda_1 = 4l$$

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{4l} = \frac{1}{4l}\sqrt{\frac{\gamma P}{\rho}} \qquad (8.1)$$

where γ is the ratio of the specific heat capacities of the air, P its pressure and ρ its density. This is the frequency of the fundamental mode, also called the **first harmonic**.

Second mode. The column holds one more node and one more antinode than in the fundamental (Figure 8.3 b): three quarter-wavelengths. With wavelength λ_2 and frequency f_2,

$$\frac{3\lambda_2}{4} = l \quad \Rightarrow \quad \lambda_2 = \frac{4l}{3}$$

$$f_2 = \frac{v}{\lambda_2} = \frac{3v}{4l} = 3f_1 \qquad (8.2)$$

This is the **third harmonic**, or the **first overtone**.

Third mode. The column holds three nodes and three antinodes (Figure 8.3 c): five quarter-wavelengths. With wavelength λ_3 and frequency f_3,

$$\frac{5\lambda_3}{4} = l \quad \Rightarrow \quad \lambda_3 = \frac{4l}{5}$$

$$f_3 = \frac{v}{\lambda_3} = \frac{5v}{4l} = 5f_1 \qquad (8.3)$$

This is the **fifth harmonic**, or the **second overtone**.

In the same way, the higher modes have $f_4 = 7f_1$, $f_5 = 9f_1$, and so on:

$$f_1 : f_2 : f_3 : \cdots = 1 : 3 : 5 : \ldots$$

Conclusions

1. The frequency of the nth mode of vibration is $f_n = (2n-1)f_1$.
2. The frequency of the mth overtone is $(2m+1)f_1$.
3. The even harmonics are missing.
4. With the even harmonics missing, the sound of a closed pipe is poorer in quality: hollower, less rich.

8.5 Stationary waves in open organ pipe

Hold a vibrating tuning fork just above one end of an open pipe. The vibration travels along the air column to the other end and is reflected there too (an open end reflects sound back into the pipe, a compression returning as a rarefaction). The wave going one way and the reflected wave superpose to form a **stationary longitudinal wave**, now with an **antinode at each open end**. Again the column can vibrate in different modes (Figure 8.4).

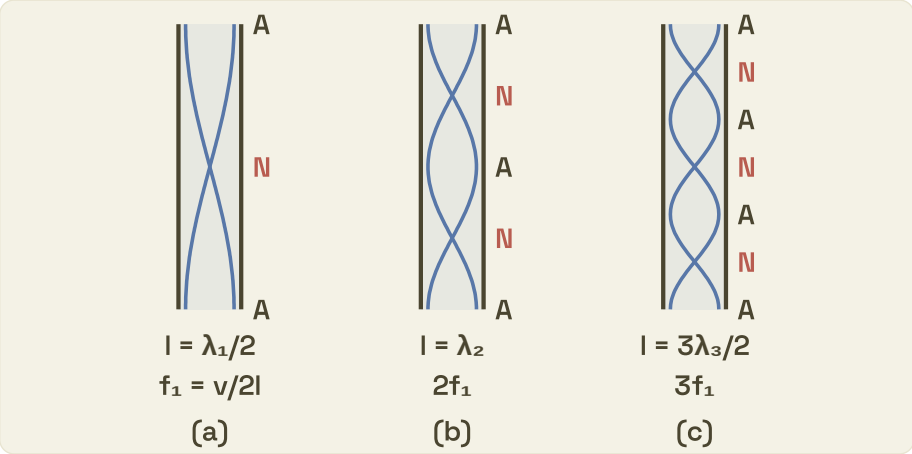


Figure 8.4 The first three modes of a pipe open at both ends. Both open ends are antinodes, A; the modes hold one, two and three half-wavelengths.

First, or fundamental, mode. The lowest possible frequency and the longest wavelength: an antinode at each end and one node between them (Figure 8.4 a). With wavelength λ_1 and frequency f_1,

Check 12

10. A string 1.5 m long, stretched under tension, vibrates with a fundamental frequency of 150 Hz. What is the speed of the wave along the string?

a 150 m/s b 225 m/s
c 300 m/s d 450 m/s

Check 13

11. The fundamental frequency of a stretched string is 100 Hz. If the tension is made 4 times as large, what will the new frequency be?

a 100 Hz b 200 Hz
c 400 Hz d 50 Hz

Check 14

12. A closed pipe resonates at 300 Hz and 500 Hz but not at 400 Hz. What is the fundamental frequency of the pipe?

a 100 Hz b 200 Hz
c 300 Hz d 500 Hz

Check 15

13. Two strings of the same length but different tensions are vibrating. The first has 4 times the tension of the second. If the fundamental frequency of the second string is f, what is that of the first?

a f b $2f$
c $4f$ d $8f$

Check 16

14. A pipe closed at one end produces a sound of wavelength 1.2 m in its fundamental mode. What is the length of the pipe, if the speed of sound is 340 m/s?

a 0.15 m b 0.30 m
c 0.60 m d 1.20 m

17.

a. Derive the expressions for the fundamental frequency and overtones in: i. a closed organ pipe; ii. an open organ pipe.
b. A closed pipe resonates with a tuning fork of frequency 256 Hz in its first harmonic. What is the length of the pipe? ($v = 343$ m/s)

18.

a. Why are only odd harmonics present in closed pipes, while all harmonics are present in open pipes?
b. Compare the harmonic content of a pipe open at both ends and one closed at one end, and explain how this affects the quality of sound.
c. An open pipe resonates in its second harmonic at 680 Hz. What is its length? ($v = 340$ m/s)
d. Calculate the frequency of the third harmonic of a closed pipe of length 0.4 m. ($v = 340$ m/s)

19.

a. Explain the concept of end correction in resonance tubes.
b. Derive the expression for the effective length of a pipe including the end correction.
c. A resonance tube has an actual length of 0.75 m. If the end correction is 0.03 m, what is the effective length of the pipe?

Exercise

Choose the correct option

Check 3

1. What is the fundamental frequency of a closed organ pipe of length l if the speed of sound in air is v?

a $v/2l$ b v/l
c $v/4l$ d $2v/l$

Check 4

2. Which of the following statements is true for an open organ pipe?

a It produces only even harmonics. b It produces only odd harmonics.
c It produces both even and odd harmonics. d It does not produce harmonics.

Check 5

3. The end correction in an organ pipe is due to:

a friction of air molecules b the effect of temperature
c the effect of a pressure difference d the vibrating air reaching beyond the pipe's end

Check 6

4. The third harmonic of a closed pipe corresponds to which overtone?

a first overtone b second overtone
c third overtone d fourth overtone

Check 7

5. If the length of a vibrating string is doubled while its tension and linear density stay the same, the frequency:

a doubles b remains the same
c becomes half d becomes four times

Check 8

6. A string fixed at both ends vibrates in its fundamental mode. What is its displacement at the center?

a zero displacement b maximum displacement
c half of the maximum displacement d twice the maximum displacement

Check 9

7. Two strings of the same material but different lengths and diameters are stretched under the same tension. Which factor most affects their vibrational frequency?

a tension b density
c length and diameter d shape of the string

Check 10

8. A flute and a guitar string produce sound of the same frequency. Which statement must be true?

a They have the same mode of vibration. b They have the same amplitude.
c They produce the same fundamental frequency. d They have the same length.

Check 11

9. A closed pipe has a fundamental frequency of 220 Hz. What will be the frequency of its first overtone?

a 220 Hz b 440 Hz
c 660 Hz d 880 Hz

Answer these questions

15. a. Define harmonics and overtones.

$$\frac{\lambda_1}{2} = l \quad \Rightarrow \quad \lambda_1 = 2l$$

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2l} = \frac{1}{2l}\sqrt{\frac{\gamma P}{\rho}} \qquad (8.4)$$

This is the **first harmonic**. It is twice the fundamental of a closed pipe of the same length: an open pipe sounds an octave higher.

Second mode. One more node and one more antinode (Figure 8.4 b): a whole wavelength. With wavelength λ_2 and frequency f_2 ,

$$\frac{2\lambda_2}{2} = l \quad \Rightarrow \quad \lambda_2 = l$$

$$f_2 = \frac{v}{\lambda_2} = \frac{2v}{2l} = 2f_1 \qquad (8.5)$$

This is the **second harmonic**, or the **first overtone**.

Third mode. Three nodes and four antinodes (Figure 8.4 c): one and a half wavelengths. With wavelength λ_3 and frequency f_3,

$$\frac{3\lambda_3}{2} = l \quad \Rightarrow \quad \lambda_3 = \frac{2l}{3}$$

$$f_3 = \frac{v}{\lambda_3} = \frac{3v}{2l} = 3f_1 \qquad (8.6)$$

This is the **third harmonic**, or the **second overtone**.

In the same way, $f_4 = 4f_1$, $f_5 = 5f_1$, and so on:

$$f_1 : f_2 : f_3 : \cdots = 1 : 2 : 3 : \ldots$$

Conclusions

1. The frequency of the nth mode of vibration is $f_n = nf_1$.
2. The frequency of the mth overtone is $(m+1)f_1$.
3. Both odd and even harmonics are present.
4. With all the harmonics present, the sound of an open pipe is richer in quality.

Check 1

A pipe closed at one end has a fundamental frequency of 150 Hz. Which of these can it **not** produce?

a 450 Hz b 600 Hz
c 750 Hz d 1050 Hz

Try both pipes, and hear them:

WORKED EXAMPLE

Example 8.1: in unison

Find the ratio of the length of a closed pipe to that of an open pipe so that the second overtone of the closed pipe is in unison with the fourth overtone of the open pipe.

Answer

Let l_1 be the length of the closed pipe and l_2 that of the open pipe. The closed pipe's second overtone is its fifth harmonic:

$$l_1 = \frac{5\lambda}{4} \quad \Rightarrow \quad \lambda = \frac{4l_1}{5}$$

The open pipe's fourth overtone is its fifth harmonic:

$$l_2 = \frac{5\lambda'}{2} \quad \Rightarrow \quad \lambda' = \frac{2l_2}{5}$$

"In unison" means the same frequency, and the speed of sound is the same, so $\lambda = \lambda'$:

$$\frac{4l_1}{5} = \frac{2l_2}{5} \quad \Rightarrow \quad \frac{l_1}{l_2} = \frac{1}{2}$$

The closed pipe must be half as long as the open pipe: $l_1 : l_2 = \mathbf{1:2}$.

WORKED EXAMPLE

Example 8.2: same length, open and closed

Compare the fundamental frequencies of an open and a closed organ pipe of the same length.

Answer

For the open pipe, $f_0 = \dfrac{v}{\lambda_0} = \dfrac{v}{2l}$. For the closed pipe, $f_0' = \dfrac{v}{\lambda_0'} = \dfrac{v}{4l}$. Dividing,

$$\frac{f_0}{f_0'} = \frac{v}{2l} \times \frac{4l}{v} = 2$$

The open pipe's fundamental is **twice** the closed pipe's: $f_0 : f_0' = 2 : 1$.

YOUR TURN

Test yourself 8.2: equal first overtones

The first overtone of a closed organ pipe has the same frequency as the first overtone of an open pipe. What is the ratio of their lengths?

Answer

Closed pipe (length l_1): first overtone = third harmonic, $\dfrac{3v}{4l_1}$. Open pipe (length l_2): first overtone = second harmonic, $\dfrac{2v}{2l_2} = \dfrac{v}{l_2}$. Equal:

$$\frac{3v}{4l_1} = \frac{v}{l_2} \quad \Rightarrow \quad \frac{l_1}{l_2} = \frac{3}{4}$$

$l_1 : l_2 = \mathbf{3 : 4}$.

8.6 End correction in pipes

So far we have assumed that the antinode forms exactly at the open end of a pipe. In fact it forms a **little way outside** the open end. The air at the open end is free to move, and its vibration carries on a short distance beyond the pipe before it spreads out into the open air. So the **acoustic (effective) length** of the pipe is greater than its physical length, and a wavelength worked out from the physical length alone is a little wrong.

The distance from the open end of the pipe to the true position of the antinode is the **end correction**, written e (sometimes x or c). To find the wavelength, the end correction is added to the length at each open end (Figure 8.5).

An open pipe has an antinode at both ends, so the end correction is added at both. For its fundamental,

$$f = \frac{v}{2(l + 2e)} \qquad (8.7)$$

A closed pipe has an antinode only at its open end, so the end correction is added once:

$$f = \frac{v}{4(l + e)} \qquad (8.8)$$

Lord Rayleigh studied the end correction with pipes of different diameters and found

$$e = 0.3d = 0.6r$$

where d is the internal diameter of the pipe and r its internal radius: the **Rayleigh end correction**. With it, (8.7) and (8.8) become

$$\text{open pipe:} \quad f = \frac{v}{2(l + 2 \times 0.3d)} = \frac{v}{2(l + 0.6d)} = \frac{v}{2(l + 1.2r)} \qquad (8.9a)$$

$$\text{closed pipe:} \quad f = \frac{v}{4(l + 0.3d)} = \frac{v}{4(l + 0.6r)} \qquad (8.9b)$$

So, for the same length, **a wider pipe gives a lower note**: the frequency falls as the internal radius grows (though it isn't inversely proportional to the radius, since r is only added to l). Organ builders use this: a wide pipe of the same length sounds slightly lower, and a pipe is trimmed to tune it.

WORKED EXAMPLE

Example 8.3: a steel wire

A steel wire 1 m long, of mass 1.0 g, is under a tension of 400 N and tied down at both ends. a. What is the wavelength of its fundamental mode of vibration? b. What is the frequency of this mode?

Answer

$l = 1$ m, $T = 400$ N, $\mu = \dfrac{1.0\text{ g}}{1\text{ m}} = 10^{-3}$ kg/m.

a. In the fundamental, $\dfrac{\lambda}{2} = l$, so $\lambda = 2l = \mathbf{2\text{ m}}$.

b.

$$f = \frac{1}{2l}\sqrt{\frac{T}{\mu}} = \frac{1}{2 \times 1}\sqrt{\frac{400}{10^{-3}}} = \frac{632.5}{2} = \mathbf{316\text{ Hz}}$$

WORKED EXAMPLE

Example 8.4: a piano string

A piano string 1.5 m long is made of steel of density 7800 kg/m³ and Young's modulus 2×10^{11} N/m². It is kept at a tension that produces an elastic strain of 1% in it. Calculate the frequency of the string when it vibrates in its second mode.

Answer

$l = 1.5$ m, $\rho = 7800$ kg/m³, $Y = 2 \times 10^{11}$ N/m², strain $= 0.01$.

The tension comes from Young's modulus: $Y = \dfrac{T/A}{\text{strain}}$, so $\dfrac{T}{A} = Y \times \text{strain}$. And $\mu = A\rho$, so

$$\frac{T}{\mu} = \frac{T}{A\rho} = \frac{Y \times \text{strain}}{\rho}$$

The fundamental:

$$f_1 = \frac{1}{2l}\sqrt{\frac{Y \times \text{strain}}{\rho}} = \frac{1}{2 \times 1.5}\sqrt{\frac{2 \times 10^{11} \times 0.01}{7800}} = \frac{506.4}{3} = 168.8\text{ Hz}$$

In the second mode, $f_2 = 2f_1 = \mathbf{337.6\text{ Hz}}$.

(The string's thickness doesn't matter: a thicker wire at the same strain is pulled harder in proportion to its extra mass.)

IN SHORT

In short

- In a pipe or string, a wave and its reflection form a stationary wave; a closed end of a pipe or a fixed end of a string is a node, an open end an antinode.
- Closed pipe: $l = \lambda/4, 3\lambda/4, 5\lambda/4 \ldots$; $f_1 = v/4l$; only odd harmonics, $f_1 : f_2 : f_3 = 1 : 3 : 5$.
- Open pipe and string: $l = \lambda/2, \lambda, 3\lambda/2 \ldots$; $f_1 = v/2l$; every harmonic, $1 : 2 : 3$.
- End correction $e = 0.3d = 0.6r$: open pipe $f = v/2(l + 2e)$, closed pipe $f = v/4(l + e)$.
- String: $v = \sqrt{T/\mu}$, $f = \dfrac{1}{2l}\sqrt{\dfrac{T}{\mu}}$; $f \propto 1/l$, $f \propto \sqrt{T}$, $f \propto 1/\sqrt{\mu}$.

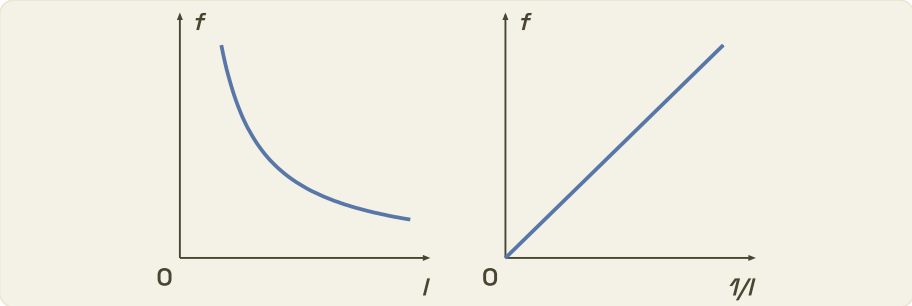


Figure 8.8 Law of length: f against l is a falling curve (a hyperbola), and f against $1/l$ a straight line through the origin.

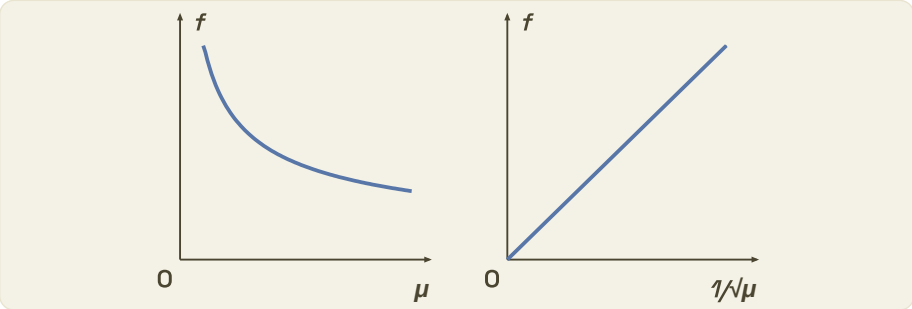


Figure 8.9 Law of mass: f against μ falls; f against $1/\sqrt{\mu}$ is a straight line through the origin.

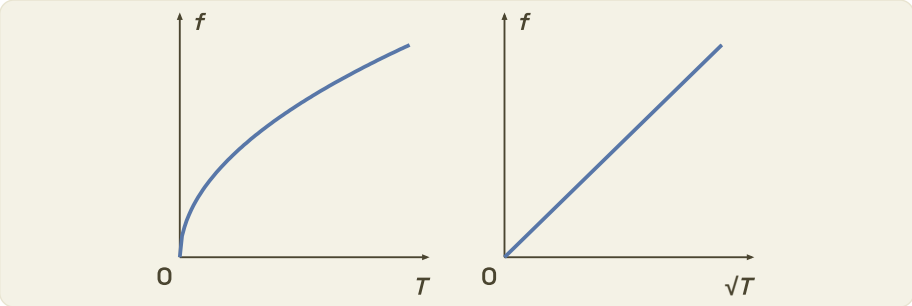


Figure 8.10 Law of tension: f against T rises ever more slowly; f against $\sqrt{T}$ is a straight line through the origin.

The laws are tested in the laboratory with a **sonometer**: a wire stretched over two bridges on a hollow wooden box, pulled tight by a weight hanging over a pulley. The bridges are moved until the wire resonates with a tuning fork (a small paper rider on the wire jumps off).

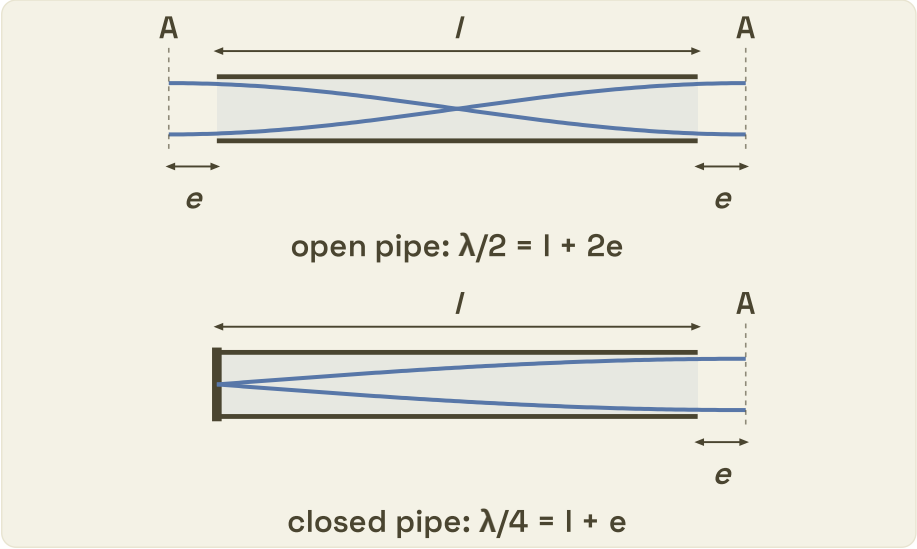


Figure 8.5 End correction. The antinode lies a distance e beyond each open end. An open pipe has two open ends, so in its fundamental $\lambda/2 = l + 2e$; a closed pipe has one, so $\lambda/4 = l + e$.

WORKED EXAMPLE

End correction in a resonance tube

A resonance tube of internal diameter 4.0 cm, closed at the bottom by water, resonates in its fundamental with a 512 Hz tuning fork when the air column is 15.4 cm long. Find the speed of sound in the air.

Answer

End correction: $e = 0.3d = 0.3 \times 4.0 = 1.2$ cm. Effective length $l + e = 15.4 + 1.2 = 16.6$ cm $= 0.166$ m.

$$\lambda = 4(l + e) = 4 \times 0.166 = 0.664 \text{ m}, \qquad v = f\lambda = 512 \times 0.664 = 340 \text{ m/s}$$

Without the end correction we would get $512 \times 4 \times 0.154 = 315$ m/s, more than 7% too low.

8.7 Velocity of transverse wave along a stretched string

A **string**, in physics, is a wire that is perfectly flexible and very thin, of the same diameter all along its length. Set vibrating, it carries transverse waves along its length. In musical instruments, the guitar, sitar and sarangi, strings are fixed at both ends and sound is made by plucking, striking or bowing them.

Let a string of length l and mass m, fixed at both ends, be under a tension T. If it is perfectly flexible, a transverse wave travels along it with speed

$$\boxed{v = \sqrt{\frac{T}{m/l}} = \sqrt{\frac{T}{\mu}}} \qquad (8.10)$$

where $\mu = \dfrac{m}{l}$ is the **mass per unit length** of the string, also called its **linear mass density** (unit kg/m). For a wire of cross-section A, radius r, diameter d and density ρ,

$$\mu = \frac{m}{l} = \frac{V\rho}{l} = \frac{Al\rho}{l} = A\rho = \pi r^2 \rho = \frac{\pi d^2}{4}\rho$$

The tension pulls a displaced piece of the string back (the elastic factor); its mass per unit length resists being moved (the inertial factor), just as in $v = \sqrt{E/\rho}$ (Unit 7).

Check 2

Two wires of the same steel are under the same tension; one has twice the diameter of the other. Waves on the thicker wire travel:

a twice as fast
b at the same speed
c at half the speed
d at a quarter of the speed

8.8 Stationary waves in stretched string

Let a string of length l, mass per unit length μ, be fixed between two rigid supports and kept under tension T (Figure 8.6). When it is plucked at any point, transverse waves travel along it towards both ends and are reflected there. The reflected and incoming waves superpose and form a **stationary transverse wave**. Since the string is fixed at both ends, **both ends are nodes**.

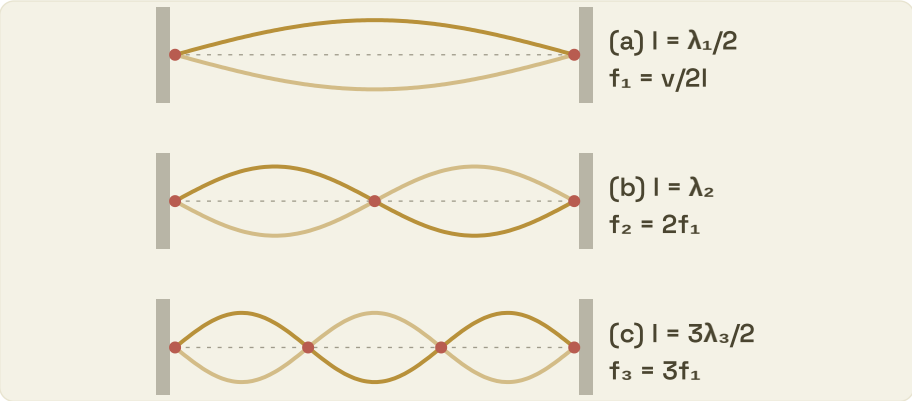


Figure 8.6 *A string fixed at both ends in its first three modes. Both ends are always nodes, and the string holds one, two or three loops, each half a wavelength long.*

First, or fundamental, mode. The string vibrates in one loop: nodes at the two ends and an antinode at the middle (Figure 8.6 a). If λ_1 is the wavelength,

$$\frac{\lambda_1}{2} = l \quad \Rightarrow \quad \lambda_1 = 2l$$

$$f_1 = \frac{v}{\lambda_1} = \frac{v}{2l} = \frac{1}{2l}\sqrt{\frac{T}{\mu}} \qquad (8.11)$$

This is the **fundamental**, or **first harmonic**.

Second mode. The string vibrates in two loops (Figure 8.6 b):

$$2\left(\frac{\lambda_2}{2}\right) = l \quad \Rightarrow \quad \lambda_2 = \frac{2l}{2} = l$$

$$f_2 = \frac{v}{\lambda_2} = \frac{2v}{2l} = 2f_1 \qquad (8.12)$$

This is the **second harmonic**, or **first overtone**.

Third mode. The string vibrates in three loops (Figure 8.6 c):

$$3\left(\frac{\lambda_3}{2}\right) = l \quad \Rightarrow \quad \lambda_3 = \frac{2l}{3}$$

$$f_3 = \frac{v}{\lambda_3} = \frac{3v}{2l} = 3f_1 \qquad (8.13)$$

This is the **third harmonic**, or **second overtone**.

In the same way, $f_4 = 4f_1$, $f_5 = 5f_1$, and so on: $f_1 : f_2 : f_3 : f_4 = 1 : 2 : 3 : 4$.

Conclusions

1. The frequency of the nth mode of vibration is $f_n = nf_1$, where $n = 1, 2, 3, \ldots$
2. The frequency of the mth overtone is $(m+1)f_1$, where $m = 1, 2, 3, \ldots$
3. Both odd and even harmonics are present.
4. With every harmonic present, a string's sound is rich in quality.

DID YOU KNOW?

One string, many notes at once

A plucked string doesn't vibrate in only one mode: it vibrates in its fundamental and many overtones together, and the mix depends on where it is plucked. Pluck a sitar string near the bridge and the upper harmonics are strong: a bright, twangy sound. Pluck it in the middle and the even harmonics, which have a node there, are missing: a softer, rounder one. Touch the string lightly at its middle while plucking and you kill the fundamental, leaving a clear note an octave higher, a **harmonic**, as players call it.



Figure 8.7 *The sarangi of Nepal's Gandharva musicians: four strings, a bow, and the player's fingernails touching the strings from the side to shorten them.*
Photo: Bishwo Ghimire, CC BY 3.0

8.9 Laws of transverse vibration of stretched string

Why does tightening a guitar string raise its note? Why does a shorter string sound higher? The fundamental frequency of a stretched string is

$$f = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$$

where l is its vibrating (resonating) length, μ its mass per unit length and T its tension. Three laws follow from it.

1. Law of length. The frequency is inversely proportional to the vibrating length, when μ and T are kept constant:

$$f \propto \frac{1}{l}$$

Example: on a guitar or a sitar, pressing the string down on a fret nearer the body shortens its vibrating length, and the note rises. Halving the length raises the note an octave.

2. Law of mass. The frequency is inversely proportional to the square root of the mass per unit length, when l and T are kept constant:

$$f \propto \frac{1}{\sqrt{\mu}}$$

Example: on a guitar the thicker strings (the bass strings) play lower notes than the thinner ones, even at the same length and tension. A heavier string vibrates more slowly.

3. Law of tension. The frequency is directly proportional to the square root of the tension, when l and μ are kept constant:

$$f \propto \sqrt{T}$$

Example: turning the tuning pegs of a violin or sitar to tighten a string raises its note. Four times the tension doubles the frequency.