Unit 8

Answers: Waves in Pipes and Strings

Quick checks

1 b A closed pipe makes only odd harmonics: 150, 450, 750, 1050 Hz (1, 3, 5, 7 times 150). 600 Hz is the fourth harmonic, an even one, so it can't sound.

2 c $\mu = \dfrac{\pi d^2}{4}\rho$: twice the diameter gives four times μ, and $v = \sqrt{T/\mu}$ is halved.

3 c A quarter of a wavelength fits: $\lambda = 4l$, $f = v/4l$. $v/2l$ is an open pipe's.

4 c With antinodes at both ends, any whole number of half-wavelengths fits, so $f_n = nf_1$ for every n.

5 d The air at an open end moves freely, and the vibration carries on a little beyond the end, so the antinode is outside the pipe.

6 a A closed pipe makes $f_1, 3f_1, 5f_1 \ldots$ The first frequency above the fundamental, the first overtone, is the third harmonic.

7 c Law of length: $f \propto 1/l$.

8 b In the fundamental, the ends are nodes and the middle is the antinode.

9 c The tension and the material (density) are the same, so what differs is l and $\mu = \pi d^2\rho/4$: $f = \dfrac{1}{2l}\sqrt{\dfrac{T}{\mu}} \propto \dfrac{1}{ld}$.

10 c A note's pitch is its fundamental frequency, so playing the same note means the same fundamental. The two still sound different because their overtones differ in strength (their quality), and their lengths, amplitudes and the way they vibrate need not match.

11 c A closed pipe's first overtone is its third harmonic: $3 \times 220 = 660$ Hz. (440 Hz would be an open pipe's.)

12 d $\lambda = 2l = 3.0$ m, so $v = f\lambda = 150 \times 3.0 = 450$ m/s.

13 b $f \propto \sqrt{T}$: $100 \times \sqrt{4} = 200$ Hz.

14 a A closed pipe's resonances are odd multiples of f_1, so neighboring ones differ by $2f_1$: $500 - 300 = 200 = 2f_1$, $f_1 = 100$ Hz. Then 300 and 500 are the 3rd and 5th harmonics, and 400 Hz, an even one, is missing, as the question says.

15 b $f \propto \sqrt{T}$ (same length, and the strings taken to be alike): $\sqrt{4} = 2$.

16 b $l = \lambda/4 = 1.2/4 = 0.30$ m. (The speed isn't needed.)

15.

a. **Harmonics** are frequencies that are whole-number multiples of the fundamental frequency, nf_1; the fundamental is the first harmonic. **Overtones** are the frequencies a system actually produces above its fundamental; the first of them is the first overtone.

b. In an **open pipe** every harmonic is produced, so the first overtone is the second harmonic ($2f_1$), the second overtone the third harmonic ($3f_1$): the mth overtone is $(m+1)f_1$. In a **closed pipe** only odd harmonics are produced, so the first overtone is the third harmonic ($3f_1$), the second overtone the fifth harmonic ($5f_1$): the mth overtone is $(2m+1)f_1$, and the second, fourth… harmonics don't exist at all.

c. $f_1 = \dfrac{v}{4l} \Rightarrow l = \dfrac{v}{4f_1} = \dfrac{340}{4 \times 200} = \mathbf{0.425\ m}$.

d. $v = \sqrt{\dfrac{T}{\mu}} = \sqrt{\dfrac{144}{0.01}} = 120$ m/s, and $f_1 = \dfrac{v}{2l} = \dfrac{120}{2 \times 0.6} = \mathbf{100\ Hz}$.

16.

a. Air blown across the mouth of a pipe (or a tuning fork held over it) sends a longitudinal wave along the air column. It is reflected at the far end: at a closed end, where the air can't move, and at an open end, where the air moves freely and the wave is sent back into the pipe. The wave going and the wave coming back have the same frequency and amplitude and travel in opposite directions, so they superpose into a stationary longitudinal wave. In a closed pipe it has a node at the closed end and an antinode at the open end, so $l = (2n-1)\lambda/4$; in an open pipe, antinodes at both ends, so $l = n\lambda/2$. Only these wavelengths build up; the column resonates at their frequencies.

b. The ends decide what can happen there. At a closed end the air can't move, so it must be a displacement node. At an open end the air moves freely (and the pressure stays at the outside air's), so it must be an antinode. The stationary wave must fit both ends at once, which allows only certain wavelengths: an odd number of quarter-wavelengths between a node and an antinode (closed pipe), a whole number of half-wavelengths between two antinodes (open pipe) or between two nodes (a string).

c. The wavelengths that fit are fixed by the length: $\lambda = 4l, 4l/3 \ldots$ (closed) or $2l, l \ldots$ (open). With the speed of sound fixed, $f = v/\lambda$, so every frequency is proportional to $1/l$: a longer pipe has longer waves and plays a lower pitch, a shorter pipe a higher one. That is why organ pipes for the low notes are long, and why lifting a finger on a flute, which shortens the air column, raises the note.

d. $f_1 = \dfrac{v}{2l} = \dfrac{340}{2 \times 0.85} = \mathbf{200\ Hz}$.

17. a. i. **Closed pipe**, length l: a node at the closed end and an antinode at the open end.

- Fundamental: $l = \lambda_1/4$, $\lambda_1 = 4l$, $f_1 = \dfrac{v}{4l}$.
- First overtone: $l = 3\lambda_2/4$, $\lambda_2 = \dfrac{4l}{3}$, $f_2 = \dfrac{3v}{4l} = 3f_1$.
- Second overtone: $l = 5\lambda_3/4$, $\lambda_3 = \dfrac{4l}{5}$, $f_3 = \dfrac{5v}{4l} = 5f_1$.
- So $f_1 : f_2 : f_3 = 1 : 3 : 5$: only odd harmonics; the mth overtone is $(2m + 1)f_1$.

ii. **Open pipe**, length l: antinodes at both ends.

- Fundamental: $l = \lambda_1/2$, $\lambda_1 = 2l$, $f_1 = \dfrac{v}{2l}$.
- First overtone: $l = \lambda_2$, $f_2 = \dfrac{v}{l} = 2f_1$.
- Second overtone: $l = 3\lambda_3/2$, $\lambda_3 = \dfrac{2l}{3}$, $f_3 = \dfrac{3v}{2l} = 3f_1$.
- So $f_1 : f_2 : f_3 = 1 : 2 : 3$: all harmonics; the mth overtone is $(m + 1)f_1$.

b. $l = \dfrac{v}{4f_1} = \dfrac{343}{4 \times 256} = \mathbf{0.335\ m}$.

18. a. A closed pipe must have a node at one end and an antinode at the other. From a node to an antinode is $\lambda/4$, and each extra node–antinode pair adds $\lambda/2$, so the length can only be an odd number of quarter-wavelengths, $l = (2n - 1)\lambda/4$, giving $f = (2n - 1)v/4l$: odd multiples of f_1. An open pipe has antinodes at both ends; from antinode to antinode is any whole number of half-wavelengths, $l = n\lambda/2$, giving $f = nv/2l$: every multiple of f_1.

b. An open pipe sounds its fundamental together with all its harmonics, $f_1, 2f_1, 3f_1, \ldots$ A closed pipe sounds only $f_1, 3f_1, 5f_1, \ldots$ The quality (timbre) of a note depends on which overtones sound with the fundamental and how strongly. With the even harmonics missing, a closed pipe's note is hollower and duller; an open pipe's, with all of them, is fuller and richer.

c. In the second harmonic an open pipe holds one whole wavelength: $l = \lambda_2 = \dfrac{v}{f_2} = \dfrac{340}{680} = \mathbf{0.5\ m}$.

(Check: its fundamental is then $340/(2 \times 0.5) = 340$ Hz, and $2 \times 340 = 680$ Hz.)

d. $f_3 = \dfrac{3v}{4l} = \dfrac{3 \times 340}{4 \times 0.4} = \dfrac{1020}{1.6} = \mathbf{637.5\ Hz}$.

19.

a. The antinode at an open end doesn't form exactly at the end, but a little outside it: the air just beyond the end keeps vibrating with the air inside before the wave spreads out. The distance from the open end to the antinode is the end correction, e. A resonance tube is a closed pipe (closed at the bottom by water), so its column behaves as if it were $l + e$ long, not l. Rayleigh found $e = 0.3d = 0.6r$, so the wider the tube, the larger the correction.

b. The effective length is the distance between the true node and antinode positions (or the two antinodes) that fix the wave. For a **closed pipe** (one open end), $l = l + e$, and in the fundamental $\frac{\lambda}{4} = l + e$, $f = \frac{v}{4(l + e)}$. For an **open pipe** (two open ends), $l = l + 2e$, and in the fundamental $\frac{\lambda}{2} = l + 2e$, $f = \frac{v}{2(l + 2e)}$. With $e = 0.3d$: $l + 0.3d$ and $l + 0.6d$.

c. A resonance tube has one open end: $l = l + e = 0.75 + 0.03 = \mathbf{0.78\ m}$.

20.

a. $v = \sqrt{T/\mu}$: the speed depends on the **tension** T (greater tension, faster waves) and the **mass per unit length** μ (heavier string, slower waves), and through $\mu = \pi r^2 \rho$, on the string's thickness and the density of its material. It does not depend on the string's length or on the frequency.

b. From $f = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$:

- **Law of length:** $f \propto 1/l$, T and μ constant.
- **Law of tension:** $f \propto \sqrt{T}$, l and μ constant.
- **Law of mass:** $f \propto 1/\sqrt{\mu}$, l and T constant.

c. A longer string has a longer fundamental wavelength ($2l$) and so a lower frequency: twice the length, half the frequency. More tension makes the waves faster and the frequency higher: four times the tension, twice the frequency. More mass per unit length makes the waves slower and the frequency lower: four times μ, half the frequency. (Example: shortening a sitar string on a fret, tightening its peg, or changing to a thinner string all raise the note.)

d. $\mu = \frac{m}{l} = \frac{0.01}{1.2} = 8.33 \times 10^{-3}$ kg/m, and

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{100}{8.33 \times 10^{-3}}} = \sqrt{1.2 \times 10^4} = \mathbf{109.5\ m/s}$$

21. a. In the third harmonic the string vibrates in **three loops**, as in Figure 8.6 (c). Each loop is half a wavelength, so $3 \times \lambda/2 = l$, $\lambda = 2l/3$. There are **4 nodes** (the two fixed ends and two points dividing the string into thirds, at $l/3$ and $2l/3$) and **3 antinodes** (the middle of each loop, at $l/6$, $l/2$ and $5l/6$). Neighboring loops move in opposite directions.

b. $v = \sqrt{\dfrac{T}{\mu}} = \sqrt{\dfrac{150}{0.005}} = \sqrt{30\,000} = 173.2$ m/s, and $f_1 = \dfrac{v}{2l} = \dfrac{173.2}{2 \times 1} = \mathbf{86.6\ Hz}$.

c. A string makes every harmonic, so its second overtone is its third harmonic: $3f_1 = 300$ Hz, $f_1 = \mathbf{100\ Hz}$.