Unit 6

Wave Motion

This unit online, with its interactives:
science-playground.com/s/12f6

Drop a pebble into Phewa Lake and rings spread out across the water. A leaf floating a few meters away bobs up and down as the rings pass, and then sits where it was: the water has not carried it anywhere. Yet something did travel across the lake, fast enough to reach the far shore and strong enough to rock a boat. On 25 April 2015 something like it spread out from the rocks under Gorkha and reached Kathmandu, about 80 km away, in well under a minute: the ground in Kathmandu shook, though not a single rock had moved from Gorkha to Kathmandu.

What travels is a **wave**: a pattern of motion handed on from one part of a medium to the next, carrying energy but leaving the medium where it was. This unit puts waves into equations. It writes down the displacement of every particle at every instant for a wave that travels (a **progressive wave**), adds two waves together by the **principle of superposition**, and shows how two waves running in opposite directions build a wave that doesn't travel at all, a **stationary wave**, the wave inside every guitar string and flute. The next two units use it all: the speed of sound (Unit 7), then pipes and strings (Unit 8).

6.1 Introduction

Wave motion describes how a disturbance or oscillation spreads through a medium or through space, carrying energy without carrying matter along. Waves explain a great deal of the natural world: sound reaching your ear, light from the Sun, the vibration of a sitar string, ripples on a pond, the shaking of an earthquake. This unit gives the mathematics of wave motion, and concentrates on two kinds: **progressive** and **stationary** waves.

A STORY FROM SCIENCE

The man who photographed a sound wave

Ernst Mach (1838–1916), an Austrian physicist and philosopher who taught for nearly thirty years in Prague, wanted to see what happens in the air around a bullet flying faster than sound. In 1886, following his plan, Peter Salcher, a teacher of physics at the naval academy in Fiume (now Rijeka, in Croatia), photographed bullets in flight in a dark room, lit by an electric spark lasting about a millionth of a second, through an arrangement that turns differences in the air's density into light and shade. Their photographs, published together in 1887, showed what no one had seen: a sharp, cone-shaped **shock wave** trailing back from the bullet's nose, the air's pressure jumping across it.

Mach showed that the angle of the cone depends only on how many times faster than sound the bullet goes. That ratio, the speed of a body divided by the speed of sound in the air around it, was later named the **Mach number** in his honor: an aircraft at Mach 2 flies twice as fast as sound. A crater on the far side of the Moon is named after him too.



Figure 6.1 *Ernst Mach (1838–1916).*
Photo: Charles Scolik, Public domain

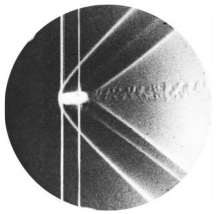


Figure 6.2 *A photograph from Mach's experiments, 1888: a bullet flying faster than sound, the cone of the shock wave spreading back from its nose.*

Photo: Ernst Mach (1838–1916), Public domain

6.2 Wave

A **wave** is a disturbance (an oscillation) that travels from one part of a medium to another, carrying energy, while each particle of the medium only vibrates about its own mean position, in **simple harmonic motion** for the waves of this unit. Sound, ripples on water, light and X-rays are examples.

A wave carries **energy** and **momentum**, but not matter. That is the difference between a wave and a moving particle:

	A moving particle (a thrown stone)	A wave (a ripple on water)
What travels	the particle itself, with its energy	energy and momentum, from particle to particle
The medium	not needed	each particle vibrates about its own place and stays there
How energy moves	carried along by one body	handed on: each particle sets the next one moving, a little later

The idea that each particle repeats the motion of the one before it, a little later, is the whole key to this unit. Section 6.4 turns it into an equation.

Check 1

A leaf floats on a pond as ripples pass it. Ignoring any wind, the leaf:

a is carried along with the ripples

b bobs up and down (and a little back and forth) about the same place

c stays perfectly still

d moves towards the stone that made the ripples

6.3 Types of waves

There are three types of waves in nature:

1. **Mechanical waves** need a material medium (a solid, a liquid or a gas) whose particles vibrate and pass the energy on: sound, ripples on water, waves on a string, seismic waves. They can't cross a vacuum. Unit 7 is about them.
2. **Electromagnetic waves** are oscillating electric and magnetic fields, and need no medium: light, radio waves, X-rays. They cross the vacuum of space at 3×10^8 m/s.
3. **Matter waves** are the waves that go with moving particles such as electrons, which the electron microscope uses.

Mechanical waves are of two kinds, by the direction the particles vibrate in:

- in a **transverse wave**, the particles vibrate at right angles to the direction the wave travels: a wave on a rope or a string, ripples on water (nearly);
- in a **longitudinal wave**, the particles vibrate along the direction the wave travels, making **compressions** and **rarefactions**: sound in air, a push along a slinky.

DID YOU KNOW?

Two waves from one earthquake

An earthquake sends out both kinds at once. The longitudinal **P waves** (primary) run through the Earth's crust at about 6 km/s; the transverse **S waves** (secondary) at about 3.5 km/s. In 2015 Kathmandu, roughly 80 km from the epicenter, felt the P waves first, a jolt, then the stronger S waves about ten seconds later. Seismologists use that gap to find how far away an earthquake was: the longer the wait, the farther the source.

Check 2

Which of these needs a material medium to travel?

a light from the Sun
b a radio signal from a satellite
c the sound of a temple bell
d an X-ray

6.4 Progressive wave

A wave that travels forward continuously in one direction, with a definite speed, carrying energy, while the particles of the medium have no net movement, is a **progressive wave** (or traveling wave). In it:

- every particle vibrates about its mean position with the **same frequency** and the **same amplitude**;
- the particles start one after another, so there is a **gradual change of phase** from each particle to the next;
- every particle passes through its mean position with the same greatest speed, but in turn, so **no particle is permanently at rest**.

Progressive waves are of two types: **transverse progressive waves**, such as ripples on water, and **longitudinal progressive waves**, such as sound.

Progressive wave equation

Let a progressive wave of frequency f, wavelength λ and amplitude a travel along the positive X-direction with speed v, starting from the origin O (Figure 6.3).

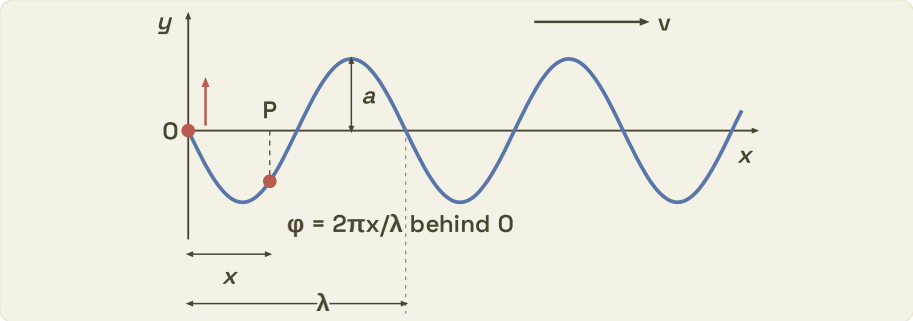


Figure 6.3 *A plane progressive wave traveling to the right, at the instant when O is at its mean position and moving up. P, a distance x along, does what O did a little earlier: its phase lags behind O's by $\phi = 2\pi x/\lambda$. Here $x = 3\lambda/8$, so $\phi = 3\pi/4$.*

1. **The particle at O.** It performs simple harmonic motion. Counting time from the instant it passes its mean position going up, its displacement is

$$y = a \sin \omega t \qquad (6.1)$$

where $\omega = 2\pi f$ is its angular frequency.

2. **The particle at P**, a distance x from O. The disturbance reaches P after a time x/v, so P does exactly what O does, only later: it lags behind O in phase by an angle ϕ.

$$y = a \sin(\omega t - \phi) \qquad (6.2)$$

3. **Phase difference from path difference.** As the wave moves one wavelength, the phase changes by one whole cycle, 2π. So a path difference of λ means a phase difference of 2π; a path difference of 1 m, $\frac{2\pi}{\lambda}$; a path difference of x,

$$\phi = \frac{2\pi}{\lambda} x$$

4. Putting this into (6.2):

$$y = a\sin\left(\omega t - \frac{2\pi}{\lambda}x\right) \qquad (6.3)$$

$$\boxed{y = a\sin(\omega t - kx)} \qquad (6.4)$$

where $k = \dfrac{2\pi}{\lambda}$ is the **wave number** (or **propagation constant**).

This is the **equation of a plane progressive wave**: it gives the displacement y of the particle at any place x at any time t. The same equation can be written in other forms:

- with $\omega = \dfrac{2\pi}{T}$ in (6.3):

$$y = a\sin\left(\frac{2\pi}{T}t - \frac{2\pi}{\lambda}x\right) = a\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \qquad (6.5)$$

- with $\omega = 2\pi f = \dfrac{2\pi v}{\lambda}$ in (6.3):

$$y = a\sin\left(\frac{2\pi v}{\lambda}t - \frac{2\pi}{\lambda}x\right) = a\sin\frac{2\pi}{\lambda}(vt - x) \qquad (6.6)$$

Equations (6.2) to (6.6) are all forms of the same progressive wave. For a wave traveling the other way, along the **negative** X-direction, P receives the disturbance **before** O, so it **leads** O by ϕ:

$$y = a\sin(\omega t + kx) \qquad (6.7)$$

and in general

$$y = a\sin(\omega t \pm kx) \qquad (6.8)$$

the minus sign for a wave towards $+x$, the plus sign for one towards $-x$.

Notes

- **The wave number** $k = 2\pi/\lambda$ is the wave's **spatial frequency**: how many radians of phase there are in each meter, so how tightly the waves are packed. Its unit is the **radian per meter** (rad/m). A larger k means a shorter wavelength, more crests in each meter, just as a larger ω means more cycles in each second.
- **Lag and lead.** For a wave towards $+x$, P lags behind O by ϕ; for a wave towards $-x$, P leads O by ϕ.
- **Speed from the equation.** Comparing (6.4) with (6.6), $\omega = kv$, so

$$v = \frac{\omega}{k} = f\lambda$$

Read ω and k off any wave equation and divide.

> CAREFUL
> **Which way does it go?**
>
> Some books write a wave as $y = a\sin(kx - \omega t)$ instead of $a\sin(\omega t - kx)$. Both travel towards $+x$: what decides the direction is whether x and t have **opposite** signs (towards $+x$) or the **same** sign (towards $-x$). The two forms differ only by a phase of π, since $\sin(kx - \omega t) = -\sin(\omega t - kx)$: the same wave, half a cycle later. To check a direction, follow a crest: in $\omega t - kx =$ constant, x must grow as t grows.

Watch the equation at work: every bead below does exactly what O does, later by the phase difference.

Characteristics of progressive wave

1. A progressive wave carries energy from one point to another, in the direction the wave travels.
2. It moves continuously in one direction; in an ideal medium (no damping, nothing in the way) it goes on unchanged.
3. Every particle vibrates with the **same amplitude** (in a medium that absorbs no energy).
4. There are **no nodes or antinodes**: no particle is permanently at rest, unlike a stationary wave.
5. Every particle of the medium oscillates about its own mean position as the wave passes; none travels along with the wave.
6. The phase changes gradually from one particle to the next.
7. It may be **transverse** (ripples on water, a wave on a rope) or **longitudinal** (sound).

16.

a. Explain how standing waves are formed in a musical instrument like a guitar.
b. Discuss the physical significance of the nodes and antinodes in the equation of a stationary wave.
c. Why do stationary waves form nodes and antinodes instead of traveling forward?
d. A string fixed at both ends vibrates in its second harmonic with a frequency of 400 Hz. The length of the string is 1.5 m. Calculate i. the speed of the wave on the string, ii. the wavelength of the wave.

17.

a. Compare the characteristics of progressive waves and stationary waves under the following heads: energy transmission; amplitude variation; phase relationship between particles; nature of nodes and antinodes.
b. A stationary wave is represented by $y(x, t) = 2A\cos(kx)\sin(\omega t)$. Identify the locations of the nodes and antinodes, and derive the expression for the distance between two successive nodes.

18.

a. Discuss the conditions necessary for the formation of stationary waves. Illustrate with examples of stationary waves in air columns and stretched strings.
b. A sound wave travels in air with speed 340 m/s. Its frequency is 1700 Hz. i. Calculate its wavelength. ii. Write the equation of the wave, taking its amplitude as 0.003 m and its direction as the positive x-direction.

A companion to your textbook, not a replacement for it: study from the book your school uses, and where the two differ, follow your teacher. Found a mistake? Please tell us.

Check 17

10. A tuning fork produces a sound wave of frequency 256 Hz. If the speed of sound in air is 340 m/s, what is the wavelength of the sound wave?

a 0.86 m b 1.33 m
c 1.45 m d 2.25 m

Check 18

11. Two progressive waves moving in opposite directions produce a stationary wave. If the amplitude of each progressive wave is 2 cm, what is the maximum amplitude of the stationary wave?

a 1 cm b 2 cm
c 4 cm d 8 cm

Check 19

12. A progressive wave is represented by $y = 0.05 \sin \pi(30t - 4x)$. What is the frequency of the wave?

a 5 Hz b 10 Hz
c 15 Hz d 30 Hz

14.

a. If two waves moving in opposite directions interfere, under what conditions will they form a stationary wave instead of canceling out?

b. Define a stationary wave. Write two examples of stationary waves.

c. Two waves traveling in opposite directions are given by $y_1 = 3\sin(4x - 200t)$ and $y_2 = 3\sin(4x + 200t)$. i. Write the equation of the stationary wave formed. ii. What is the distance between two consecutive nodes?

15.

a. State the principle of superposition of waves.

b. How does the principle of superposition help in the formation of stationary waves?

c. Derive the equation of a stationary wave formed by the superposition of two identical progressive waves traveling in opposite directions.

d. A standing wave is set up on a string of length 1.2 m with 3 antinodes. i. What is the wavelength? ii. If the frequency is 120 Hz, find the wave speed.

Check 3

A wave is $y = 0.03 \sin(40t + 5x)$ in SI units. Which statement is right?

a it travels towards $+x$ at 8 m/s b it travels towards $-x$ at 8 m/s
c it travels towards $-x$ at 200 m/s d its wavelength is 5 m

Check 4

Two points on a progressive wave of wavelength 60 cm are 15 cm apart. Their phase difference is:

a π b $\pi/2$
c $\pi/4$ d 2π

6.5 Principle of superposition of waves

When two or more waves arrive at the same place, each particle there is pushed by all of them at once, and the medium takes a new shape: a new wave. This is the **superposition of waves**.

The **principle of superposition** states: *when two or more waves overlap in a region of space, the resultant displacement of a particle is the vector sum of the displacements it would have from each wave alone.* For waves whose displacements lie along the same line, the vector sum is an ordinary sum with signs:

$$y = y_1 + y_2 + y_3 + \dots$$

The waves pass through one another and come out unchanged; only where they overlap do their displacements add. Superposition gives different results in different cases:

1. two waves of the same frequency traveling in the **same direction**: **interference** (Unit 11, with light);
2. two waves of the same frequency traveling in **opposite directions**: a **stationary wave** (below);
3. two waves of **slightly different frequencies** traveling in the same direction: **beats**, a sound that grows loud and soft again and again (not in your course).

Check 5

A crest of height 3 cm meets a trough of depth 2 cm traveling the other way on the same rope. At the instant they exactly overlap, the rope there is displaced by:

a 5 cm b 1 cm, upward
c 1 cm, downward d zero

Formation of stationary wave

When two progressive waves of the **same amplitude, speed and frequency** travel through a medium in exactly **opposite directions** and superpose, they form a **stationary wave** (or standing wave). It is called stationary because the wave pattern doesn't travel either way, and no energy is carried past any point: the energy stays trapped, swapping between kinetic and potential energy, in each segment.

The easiest way to make the second wave is to reflect the first: a wave sent along a string fixed at its far end comes back, and the two overlap all along the string.

- **Transverse stationary waves** form on the strings of a guitar, sitar, sarangi, piano and sonometer.
- **Longitudinal stationary waves** form in the air columns of organ pipes, flutes and the resonance tube.

In a stationary wave each particle has **its own amplitude**, and so its own greatest velocity and acceleration:

- At some points the particles are **permanently at rest**, with zero displacement at every instant: these are the **nodes** (N).
- Midway between two nodes the displacement swings to its largest values: the **antinodes** (A).
- The amplitude grows from zero at a node to its greatest at the antinode, and falls to zero again at the next node.
- All the particles between two neighboring nodes (one **loop** or segment) move **in phase**: they all go up together and all reach their extremes together. The particles of the next loop move **in opposite phase** (a phase difference of π): when one loop goes up, its neighbors go down.
- All the particles pass through their mean positions at the same instant, each then at its own greatest speed, and the whole string is straight for that instant.
- In a longitudinal stationary wave, the pressure and density change **most at the nodes** and **least (not at all) at the antinodes**: at a node the layers of air on its two sides move towards it together, then away from it together, squeezing and stretching it, while at an antinode the air moves freely back and forth with no squeeze.

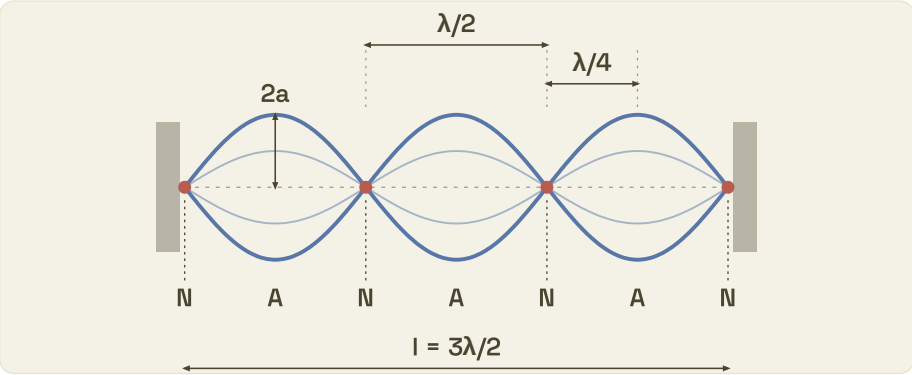


Figure 6.4 *A stationary wave on a string fixed at both ends, drawn at five instants. The nodes, N, never move; the antinodes, A, swing furthest, to ±2a, where a is the amplitude of each of the two waves that make it. Nodes are λ/2 apart; a node and the next antinode, λ/4. This string is 3λ/2 long, three loops.*

Play with it: two waves run through each other, and their sum stays put.

Stationary wave equation

Let two progressive waves of the same frequency f, wavelength λ and amplitude a travel through a medium in opposite directions with the same speed v:

$$y_1 = a\sin(\omega t - kx) \qquad (6.9)$$

$$y_2 = a\sin(\omega t + kx) \qquad (6.10)$$

1. By the principle of superposition, the resultant displacement is

$$y = y_1 + y_2 = a\left[\sin(\omega t - kx) + \sin(\omega t + kx)\right]$$

2. Use $\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$, with $C = \omega t - kx$ and $D = \omega t + kx$: $\dfrac{C+D}{2} = \omega t$ and $\dfrac{C-D}{2} = -kx$, and $\cos(-kx) = \cos kx$. So

$$y = 2a\cos kx\,\sin\omega t$$

3. Writing $A = 2a\cos kx$,

$$\boxed{y = A\sin\omega t, \qquad A = 2a\cos kx} \qquad (6.11)$$

This is the **equation of a stationary wave**. Read it as two pieces:

- $\sin\omega t$: every particle performs simple harmonic motion with the **same frequency** as the two waves that made it;
- $A = 2a\cos kx$: the **amplitude depends on the place** x, from $2a$ down to 0.

There is no $(\omega t - kx)$ in it, nothing that moves along with time: that is why the pattern doesn't travel.

Positions of the antinodes

The amplitude is greatest, $|A| = 2a$, where $\cos kx = \pm 1$:

$$\frac{2\pi}{\lambda}x = n\pi \quad\Rightarrow\quad x = \frac{n\lambda}{2}, \qquad n = 0, 1, 2, 3, \ldots$$

So the antinodes are at $x = 0, \dfrac{\lambda}{2}, \lambda, \dfrac{3\lambda}{2}, 2\lambda, \ldots$, where the two waves differ in phase by $0, 2\pi, 4\pi, \ldots$ (phase difference $2kx$). Two neighboring antinodes are

$$\frac{n\lambda}{2} - \frac{(n-1)\lambda}{2} = \frac{\lambda}{2} \text{ apart.}$$

Positions of the nodes

The amplitude is zero where $\cos kx = 0$:

$$\frac{2\pi}{\lambda}x = (2n+1)\frac{\pi}{2} \quad\Rightarrow\quad x = (2n+1)\frac{\lambda}{4}, \qquad n = 0, 1, 2, 3, \ldots$$

So the nodes are at $x = \dfrac{\lambda}{4}, \dfrac{3\lambda}{4}, \dfrac{5\lambda}{4}, \dfrac{7\lambda}{4}, \ldots$, where the two waves are in opposite phase $(\pi, 3\pi, 5\pi, \ldots)$ and cancel. Two neighboring nodes are

Exercise

Choose the correct option

Check 8

1. Which of the following is true for a progressive wave?

a Particles remain stationary while the wave propagates.
b Energy is transferred without actual movement of matter.
c Nodes and antinodes form alternately.
d The frequency of the wave changes continuously.

Check 9

2. Which equation represents a progressive wave moving in the positive x-direction?

a $y = a\sin(\omega t - kx)$
b $y = a\sin(\omega t + kx)$
c $y = a\cos(\omega t + kx)$
d $y = a\cot(\omega t + kx)$

Check 10

3. What is the necessary condition for the formation of stationary waves?

a two waves traveling in the same direction with the same amplitude
b two waves traveling in opposite directions with the same amplitude and frequency
c two waves having different wavelengths and frequencies
d two waves having the same frequency but different amplitudes

Check 11

4. In a progressive wave, what does the velocity of the wave depend on?

a only the amplitude of the wave
b the medium through which the wave propagates
c only the frequency of the wave
d the phase difference between two points

Check 12

5. If the frequency of a progressive wave is doubled while its wavelength remains constant, what happens to the wave velocity?

a It remains the same.
b It is doubled.
c It is halved.
d It becomes four times.

Check 13

6. A string is fixed at both ends and a stationary wave is formed on it. What is the distance between two consecutive nodes?

a λ/4
b λ/2
c λ
d 2λ

Check 14

7. A wave is given by $y = 0.02\sin(20\pi t - 4\pi x)$ in SI units. What is the wave velocity?

a 4 m/s
b 5 m/s
c 10 m/s
d 20 m/s

Check 15

8. A wave has a frequency of 500 Hz and a velocity of 340 m/s. What is its wavelength?

a 0.58 m
b 0.68 m
c 0.75 m
d 0.85 m

Check 16

9. A string of length 2 m is fixed at both ends and vibrates in two segments. What is the wavelength of the standing wave?

a 1 m
b 2 m
c 4 m
d 0.5 m

Answer these questions

IN SHORT

In short

- A wave carries energy and momentum from place to place, while each particle only vibrates about its own mean position.
- A progressive wave towards $+x$: $y = a\sin(\omega t - kx) = a\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) = a\sin\frac{2\pi}{\lambda}(vt - x)$, with $k = 2\pi/\lambda$ and $v = \omega/k = f\lambda$; towards $-x$, $+kx$.
- Phase difference $\phi = \frac{2\pi}{\lambda} \times$ path difference.
- Superposition: the resultant displacement is the sum of the separate displacements.
- Two equal waves in opposite directions make a stationary wave, $y = 2a\cos kx\,\sin\omega t$: antinodes (amplitude $2a$) at $x = n\lambda/2$, nodes at $x = (2n+1)\lambda/4$; nodes $\lambda/2$ apart, node to antinode $\lambda/4$.
- A fixed (closed) end is a displacement node; a free (open) end is an antinode.

$$(2n+1)\frac{\lambda}{4} - [2(n-1)+1]\,\frac{\lambda}{4} = \frac{\lambda}{2}\ \text{apart,}$$

and a node and the nearest antinode $(2n+1)\frac{\lambda}{4} - \frac{n\lambda}{2} = \frac{\lambda}{4}$ apart.

So **nodes and antinodes alternate, $\lambda/4$ apart, and two successive nodes, or two successive antinodes, are $\lambda/2$ apart.** Measure the distance between two nodes and you have half the wavelength: that is how the wavelength of sound is found in a resonance tube.

DID YOU KNOW?

Measure the speed of light with a chocolate bar

A microwave oven is a metal box, and its microwaves, reflected off the walls, form stationary waves inside: that is why ovens have a turntable, to carry the food through the hot spots. Take the turntable out (with an adult; turn the plate it stands on upside down so nothing turns), lay a bar of chocolate on it and heat it for about 15 to 20 seconds. It melts in spots at the antinodes, about 6 cm apart, which is $\lambda/2$. The oven's frequency is printed on its back, usually 2450 MHz. Then $c = f\lambda = 2.45 \times 10^9 \times 0.12 \approx 3 \times 10^8$ m/s. (Eat the experiment afterwards.)

Boundary cases during the formation of stationary wave

A stationary wave is made by a wave and its reflection, so what happens at the end where it reflects decides whether that end is a node or an antinode. Take $x = 0$ at the reflecting end.

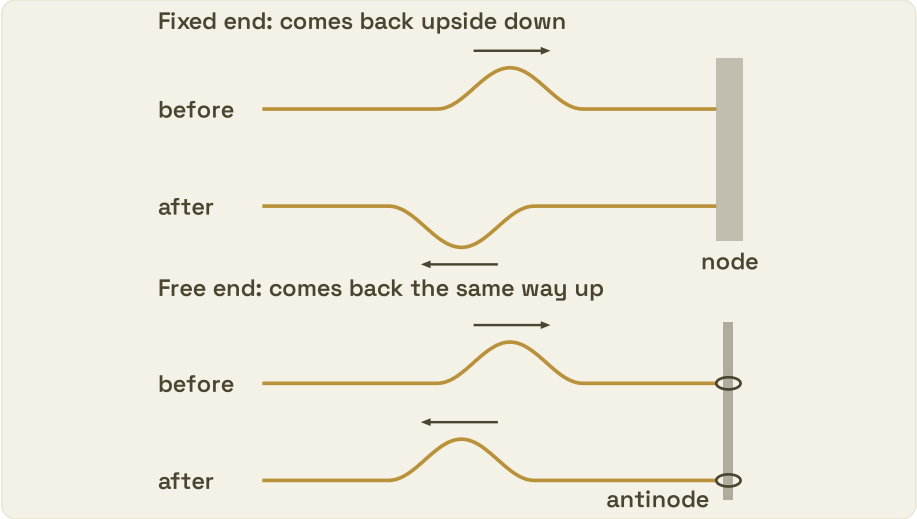


Figure 6.5 *A pulse reflected at the two kinds of end. At a fixed end (the rope tied to a wall) it comes back upside down, and the end, which can't move, is a node. At a free end (a light ring that slides on a smooth pole) it comes back the same way up, and the end swings most: an antinode.*

Open (free) boundary. The end is free to move: the free end of a rope, the open end of a pipe. At the open end of a pipe a compression comes back as a rarefaction and a rarefaction as a compression, but the **displacement** of the particles is reflected without a change of phase. So take

$$y_1 = a\sin(\omega t - kx), \qquad y_2 = a\sin(\omega t + kx)$$

and, as above, $y = 2a\cos kx\,\sin\omega t$: at the boundary $x = 0$ the amplitude is $2a$, the greatest. **An antinode forms at an open (free) end.**

Closed (fixed) boundary. The end can't move: a string tied to a peg, the closed end of a pipe. A compression comes back as a compression and a rarefaction as a rarefaction, but the **displacement** is reflected with a phase change of π (the pulse comes back upside down). So take

$$y_1 = a\sin(\omega t - kx), \qquad y_2 = a\sin(\omega t + kx + \pi) = -a\sin(\omega t + kx)$$

Then, with $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$,

$$y = a[\sin(\omega t - kx) - \sin(\omega t + kx)] = 2a\cos\omega t\,\sin(-kx) = -2a\sin kx\,\cos\omega t$$

The amplitude is $-2a\sin kx$, which is zero at $x = 0$. **A node forms at a closed (fixed) end.**

CAREFUL

Displacement nodes are pressure antinodes

In a pipe, "node" and "antinode" always mean the particles' **displacement** unless a book says otherwise. A displacement node (the closed end) is where the pressure changes most, a **pressure antinode**; a displacement antinode (the open end) is where the pressure hardly changes, since the air there joins the outside air: a **pressure node**. Exam answers about pipes are expected in displacement terms: node at a closed end, antinode at an open end.

Characteristics of stationary wave

1. Its frequency is the frequency of the two progressive waves that make it.
2. The pattern doesn't move: the wave appears to stand still, with points of permanent constructive interference (antinodes) and destructive interference (nodes).
3. It forms in any bounded medium: on stretched strings (transverse stationary waves), in air columns (pipes and resonance tubes, longitudinal stationary waves), on drum skins and in rooms.
4. Particles between two neighboring nodes are in phase; particles in neighboring loops differ in phase by π.
5. The velocity of a particle at a node is always zero; elsewhere each particle has its own greatest velocity, largest at the antinodes.
6. It carries no energy along the medium.

	Progressive wave	Stationary wave
Energy	carried along in the direction of travel	not carried past any point; stays in each loop
Amplitude	the same for every particle	changes with place: 0 at nodes, $2a$ at antinodes
Phase	changes gradually from particle to particle	the same for all particles in one loop; opposite in the next loop
Nodes and antinodes	none; no particle is always at rest	nodes always at rest, $\lambda/2$ apart, with antinodes midway
The pattern	moves forward with speed v	stays in place
Shape of the medium	never straight everywhere at once	straight everywhere twice in each period
Equation	$y = a\sin(\omega t \pm kx)$	$y = 2a\cos kx\,\sin\omega t$

WORKED EXAMPLE

Example 6.1: reading a wave equation

The equation $y = 0.1\sin\left(200\pi t - \dfrac{20\pi x}{17}\right)$ describes a plane progressive wave, with y in mm, t in s and x in m, measured from a fixed origin O. Work out: i. its frequency; ii. its wavelength; iii. its speed; iv. the phase difference, in radians, between points 0.25 m and 1.10 m from O.

Answer

Compare with $y = a\sin(\omega t - kx)$: amplitude $a = 0.1$ mm $= 10^{-4}$ m, $\omega = 200\pi$ rad/s, $k = \dfrac{20\pi}{17}$ rad/m.

i. $f = \dfrac{\omega}{2\pi} = \dfrac{200\pi}{2\pi} = \mathbf{100\ Hz}$.

ii. $\lambda = \dfrac{2\pi}{k} = 2\pi \times \dfrac{17}{20\pi} = \mathbf{1.7\ m}$.

iii. $v = f\lambda = 100 \times 1.7 = \mathbf{170\ m/s}$ (check: $\omega/k = 200\pi \times 17/(20\pi) = 170$ m/s).

iv. Path difference $x = 1.10 - 0.25 = 0.85$ m, so

$$\phi = \frac{2\pi}{\lambda}x = \frac{2\pi}{1.7} \times 0.85 = \pi \text{ rad}$$

The two points are half a wavelength apart: they always move in opposite directions.

WORKED EXAMPLE

Example 6.2: amplitude at a point

Stationary waves are set up by the superposition of $y_1 = 0.05\sin(5\pi t - x)$ and $y_2 = 0.05\sin(5\pi t + x)$, where x and y are in meters and t in seconds. Find the amplitude of the particle at $x = 1$ m.

Answer

By superposition,

$$y = y_1 + y_2 = 0.05 \times 2\sin\frac{(5\pi t - x) + (5\pi t + x)}{2}\cos\frac{(5\pi t + x) - (5\pi t - x)}{2} = 0.1\cos x\,\sin 5\pi t$$

So the amplitude at x is $0.1\cos x$, with x in **radians** (here $k = 1$ rad/m). At $x = 1$ m:

$$A = 0.1\cos(1\text{ rad}) = 0.1\cos 57.3^\circ = 0.1 \times 0.5403 = 0.054\text{ m}$$

The amplitude there is **0.054 m** (5.4 cm), a little over half the greatest amplitude, 0.1 m.

YOUR TURN

Test yourself 6.7: a wave on a coil spring

A transverse wave travels along a coil spring: $y = 4.0\sin\pi(0.01x - 2.0t)$, with y and x in cm and t in s. Find: i. the amplitude; ii. the wavelength; iii. the initial phase at the origin; iv. the speed; v. the frequency of the wave.

Answer

Write it as $y = 4.0\sin(0.01\pi x - 2\pi t)$: $a = 4.0$ cm, $k = 0.01\pi$ rad/cm, $\omega = 2\pi$ rad/s. The signs of x and t are opposite, so it travels towards $+x$.

i. Amplitude = **4.0 cm**.

ii. $\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{0.01\pi} = 200$ cm = **2.0 m**.

iii. At the origin, $y = 4.0\sin(-2\pi t)$. The answer depends on the form the phase is measured in. In the question's own form, $a\sin(kx - \omega t + \phi_0)$, the phase at $x = 0$, $t = 0$ is $\phi_0 = \mathbf{0}$. In this unit's form, $a\sin(\omega t - kx + \phi_0)$, $y = 4.0\sin(2\pi t + \pi)$, so $\phi_0 = \boldsymbol{\pi}$ **rad**. Either way the motion is the same: at $t = 0$ the origin is at its mean position, moving down. (Say which form you measured it in.)

iv. $v = \dfrac{\omega}{k} = \dfrac{2\pi}{0.01\pi} = 200$ cm/s = **2.0 m/s**.

v. $f = \dfrac{\omega}{2\pi} = \mathbf{1.0\ Hz}$ (check: $f\lambda = 1.0 \times 2.0 = 2.0$ m/s).

Check 6

Two waves of amplitude 3 mm and wavelength 40 cm run in opposite directions along a string and form a stationary wave. A particle 10 cm from an antinode has an amplitude of:

a 6 mm
b 3 mm
c 0
d 1.5 mm

Check 7

In a stationary wave, two particles in neighboring loops (on either side of a node) move:

a in phase
b in opposite phase
c with a phase difference of $\pi/2$
d with a phase difference that depends on their distance apart