Unit 6

Answers: Wave Motion

Quick checks

1 b The ripples carry energy across the pond, but the water, and the leaf on it, only move about their own places. A water wave moves each bit of water round a small loop, so the leaf goes up and down and slightly to and fro, and ends where it began.

2 c Sound is a mechanical wave: the air's molecules pass the vibration along. The other three are electromagnetic waves, which cross the vacuum of space.

3 b t and x have the same sign, so the wave goes towards $-x$. Its speed is $\omega/k = 40/5 = 8$ m/s, and its wavelength $2\pi/5 \approx 1.26$ m. 200 comes from multiplying ω and k instead of dividing.

4 b $\phi = \dfrac{2\pi}{\lambda}x = \dfrac{2\pi}{60} \times 15 = \dfrac{\pi}{2}$: a quarter of a wavelength is a quarter of a cycle. When one is at a crest, the other is passing through the middle.

5 b By superposition, $y = (+3) + (-2) = +1$ cm. After they have passed, each goes on unchanged: a 3 cm crest and a 2 cm trough, as if they had never met.

6 c 10 cm is $\lambda/4$: a quarter of a wavelength from an antinode is a node, where the amplitude is always 0. 6 mm is the amplitude at the antinode itself ($2a$).

7 b $A = 2a\cos kx$ changes sign at each node, so the particles of neighboring loops always move in opposite directions: phase difference π. Within one loop all are in phase, whatever their distance. A phase difference that grows with distance belongs to a progressive wave.

8 b A progressive wave carries energy forward while each particle only vibrates about its own place. Its particles do move (they vibrate), it has no nodes or antinodes (those belong to stationary waves), and its frequency is fixed by the source.

9 a Opposite signs of t and x mean a wave towards $+x$: a crest, $\omega t - kx =$ constant, moves to larger x as t grows. The $+kx$ forms travel towards $-x$, and a cotangent isn't a wave of finite amplitude at all.

10 b A stationary wave needs two waves of the same frequency and amplitude running in opposite directions. Waves in the same direction interfere or make beats; different amplitudes leave nodes that aren't still.

11 b A mechanical wave's speed is set by the medium's elasticity and inertia, $v = \sqrt{E/\rho}$ (Unit 7). The source sets the frequency; the wavelength then adjusts, $\lambda = v/f$.

12 b $v = f\lambda$: twice f with the same λ gives twice v. (In a real medium it can't happen that way: the medium fixes v, so doubling f halves λ. The question is about the formula.)

13 **b** Nodes are at $x = (2n+1)\lambda/4$, so neighbors are $\lambda/2$ apart. $\lambda/4$ is the distance from a node to the next antinode.

14 **b** $v = \dfrac{\omega}{k} = \dfrac{20\pi}{4\pi} = 5$ m/s. (Or $f = 10$ Hz, $\lambda = 2\pi/4\pi = 0.5$ m, $v = 10 \times 0.5 = 5$ m/s.)

15 **b** $\lambda = \dfrac{v}{f} = \dfrac{340}{500} = 0.68$ m.

16 **b** Each segment (loop) is $\lambda/2$ long. Two segments: $2 \times \lambda/2 = \lambda = 2$ m.

17 **b** $\lambda = \dfrac{v}{f} = \dfrac{340}{256} = 1.33$ m.

18 **c** At an antinode the two add up fully: $2a = 2 \times 2 = 4$ cm.

19 **c** $\omega = 30\pi$ rad/s, so $f = \dfrac{\omega}{2\pi} = \dfrac{30\pi}{2\pi} = 15$ Hz. 30 Hz forgets the 2π.

13. a. A **progressive wave** is a wave that travels continuously in one direction with a definite speed, carrying energy, while the particles of the medium vibrate about their mean positions with the same amplitude and frequency, each a little later than the one before.

Derivation: let the wave travel along $+x$ from O with amplitude a, wavelength λ, frequency f, speed v.

- The particle at O moves in SHM: $y = a\sin\omega t$, $\omega = 2\pi f$.
- A particle P at distance x does the same, but lags by a phase ϕ: $y = a\sin(\omega t - \phi)$.
- A path difference λ gives a phase difference 2π, so a path difference x gives $\phi = \dfrac{2\pi}{\lambda}x$.
- Hence $y = a\sin\left(\omega t - \dfrac{2\pi}{\lambda}x\right) = a\sin(\omega t - kx)$, with $k = 2\pi/\lambda$.
- With $\omega = 2\pi/T$: $y = a\sin 2\pi\left(\dfrac{t}{T} - \dfrac{x}{\lambda}\right)$.
- With $\omega = 2\pi v/\lambda$: $y = a\sin\dfrac{2\pi}{\lambda}(vt - x)$.
- For a wave towards $-x$, P leads O: $y = a\sin(\omega t + kx)$.

b. Sound traveling through air; ripples spreading on a pond (also: a wave sent along a rope, light from the Sun, seismic waves).

c. y: the displacement of the particle at position x at time t. A: the amplitude, the greatest displacement of each particle. $k = 2\pi/\lambda$: the wave number, the phase change per meter, which sets the wavelength. $\omega = 2\pi f$: the angular frequency, the phase change per second, which sets how fast each particle vibrates. x and t: where and when. $(kx - \omega t)$: the phase; since x and t have opposite signs, the wave travels towards $+x$ with speed $v = \omega/k$.

Other forms: $y = A\sin 2\pi\left(\dfrac{x}{\lambda} - \dfrac{t}{T}\right) = A\sin\dfrac{2\pi}{\lambda}(x - vt)$. (It is the same as $-A\sin(\omega t - kx)$: the wave of part a, half a cycle shifted.)

d. Comparing, $A = 0.02$ m, $k = 5$ rad/m, $\omega = 100$ rad/s.

i. Amplitude = **0.02 m**.

ii. $\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{5} = $ **1.26 m**.

iii. $f = \dfrac{\omega}{2\pi} = \dfrac{100}{2\pi} = $ **15.9 Hz**.

iv. $v = \dfrac{\omega}{k} = \dfrac{100}{5} = $ **20 m/s** (check: $15.9 \times 1.26 = 20$).

14.

a. Two waves running in opposite directions never cancel everywhere for good: as they move through each other they cancel only at certain fixed points. They form a steady stationary wave when they have the **same frequency** (so, in the same medium, the same wavelength and speed), the **same amplitude**, travel along the **same line** in **opposite directions**, and overlap continuously, as a wave and its reflection do on a string or in a pipe.

b. A **stationary wave** is the wave formed by the superposition of two identical progressive waves (same amplitude, frequency and speed) traveling in opposite directions; its pattern doesn't travel, and it has fixed nodes and antinodes. Examples: the vibrating string of a sitar or guitar; the air column of a flute or an organ pipe.

c. i. $y = y_1 + y_2 = 3 \times 2\sin\dfrac{(4x - 200t) + (4x + 200t)}{2}\cos\dfrac{(4x - 200t) - (4x + 200t)}{2} = 6\sin 4x\cos(-200t)$, so

$$y = 6\sin(4x)\cos(200t)$$

ii. Nodes are where $\sin 4x = 0$: $4x = n\pi$, $x = n\pi/4$. Consecutive nodes are $\dfrac{\pi}{4} =$ **0.785 m** apart. (Check: $\lambda = 2\pi/k = 2\pi/4 = 1.57$ m, and $\lambda/2 = 0.785$ m.)

15.

a. When two or more waves overlap in a region, the resultant displacement of each particle is the vector sum of the displacements it would have from each wave alone: $y = y_1 + y_2 + \dots$

b. A wave and its reflection travel in opposite directions through the same medium. At every point and every instant the displacement is the sum of the two. At some points (nodes) the two are always equal and opposite, so the sum is always zero; at others (antinodes) they always add to the greatest swing, $2a$. Superposition is what turns two traveling waves into one pattern that stays in place.

c. Let $y_1 = a\sin(\omega t - kx)$ and $y_2 = a\sin(\omega t + kx)$. By superposition, $y = a[\sin(\omega t - kx) + \sin(\omega t + kx)] = 2a\sin\omega t\cos kx$ (using $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$ and $\cos(-kx) = \cos kx$). So $y = A\sin\omega t$ with $A = 2a\cos kx$: every particle moves in SHM of the same frequency, with an amplitude that depends on its place, greatest ($2a$) at $x = n\lambda/2$ (antinodes) and zero at $x = (2n+1)\lambda/4$ (nodes).

d. i. A string fixed at both ends has nodes at its ends; 3 antinodes means 3 loops, each $\lambda/2$:

$$3 \times \frac{\lambda}{2} = 1.2 \quad \Rightarrow \quad \lambda = \frac{2 \times 1.2}{3} = \mathbf{0.8\ m}$$

ii. $v = f\lambda = 120 \times 0.8 = \mathbf{96\ m/s}$.

(An answer of 1.2 m and 144 m/s counts two loops, three nodes, not three antinodes: with both ends fixed, three antinodes always means three loops.)

16.

a. When a guitar string is plucked, transverse waves run along it in both directions and reflect at the two fixed ends (the nut and the bridge), coming back upside down. The waves going each way overlap and superpose. Only the waves whose half-wavelengths fit a whole number of times into the string's length build up into steady stationary waves; the others die out. The fixed ends are nodes. The string's vibration shakes the bridge and the hollow body, which makes the air vibrate loudly at the same frequencies.

b. In $y = 2a\cos kx \sin \omega t$, the amplitude $2a\cos kx$ depends on position. At the **nodes** ($\cos kx = 0$) the two waves always cancel: the particles never move, and (in a pipe) the pressure changes most there. At the **antinodes** ($\cos kx = \pm 1$) the two waves always add up: the particles swing with the greatest amplitude, $2a$. The nodes fix the pattern in place, and the distance between them, $\lambda/2$, gives the wavelength.

c. Because the two waves carry equal energy in opposite directions, nothing is carried on balance. At every point the phase difference between the two waves stays the same for all time (it is $2kx$, with no t in it), so each point always gets the same kind of superposition: always canceling at a node, always reinforcing at an antinode. The pattern of where they cancel and where they add can't move, so the wave appears to stand still.

d. In the second harmonic the string has two loops: $2 \times \dfrac{\lambda}{2} = l$, so $\lambda = l = \mathbf{1.5\ m}$.

i. $v = f\lambda = 400 \times 1.5 = \mathbf{600\ m/s}$.

ii. $\lambda = \mathbf{1.5\ m}$.

17. a.

	Progressive wave	Stationary wave
Energy transmission	energy is carried along in the direction of travel	no energy is carried past any point; it stays in each loop
Amplitude variation	every particle has the same amplitude	amplitude changes with position, from 0 at nodes to $2A$ at antinodes
Phase relationship	the phase changes gradually from particle to particle ($2\pi x/\lambda$)	all particles in a loop are in phase; neighboring loops are in opposite phase
Nodes and antinodes	none: every particle vibrates	nodes (always at rest) and antinodes (greatest amplitude) alternate, $\lambda/4$ apart

b. The amplitude at x is $2A\cos kx$.

Antinodes: $\cos kx = \pm 1$: $kx = n\pi$, $x = \dfrac{n\pi}{k} = \dfrac{n\lambda}{2}$: $x = 0, \lambda/2, \lambda, 3\lambda/2, \ldots$

Nodes: $\cos kx = 0$: $kx = (2n+1)\dfrac{\pi}{2}$, $x = (2n+1)\dfrac{\lambda}{4}$: $x = \lambda/4, 3\lambda/4, 5\lambda/4, \ldots$

Distance between successive nodes: $x_{n+1} - x_n = (2n+3)\dfrac{\lambda}{4} - (2n+1)\dfrac{\lambda}{4} = \dfrac{2\lambda}{4} = \dfrac{\lambda}{2}$.

18. a. Conditions: two progressive waves of the **same frequency and wavelength**, the **same amplitude** (or nearly), traveling along the same line in **opposite directions** in the same medium, overlapping continuously. In practice the medium is bounded, so the wave is reflected at its ends and the wave and its reflection form the stationary wave; it builds up strongly only when the length fits a whole number of loops (resonance).

- **Air columns:** in a flute or an organ pipe, a longitudinal wave travels down the air column, reflects at the far end and comes back. A closed end is a displacement node and an open end an antinode; a pipe closed at one end fits an odd number of quarter-wavelengths ($l = \lambda/4, 3\lambda/4, \ldots$), one open at both ends a whole number of half-wavelengths.
- **Stretched strings:** on a sitar or a sonometer wire, transverse waves reflect at the two fixed ends, which are nodes; the string fits a whole number of loops, $l = \lambda/2, \lambda, 3\lambda/2, \ldots$

b. i. $\lambda = \dfrac{v}{f} = \dfrac{340}{1700} = \mathbf{0.2\ m}$.

ii. $\omega = 2\pi f = 2\pi \times 1700 = 3400\pi$ rad/s; $k = \dfrac{2\pi}{\lambda} = \dfrac{2\pi}{0.2} = 10\pi$ rad/m. Towards $+x$:

$$y = 0.003 \sin(3400\pi t - 10\pi x)$$

in meters and seconds.