Unit 3

# Fluid Mechanics

This unit online, with its interactives:
science-playground.com/s/12f3

Pilgrims at Muktinath, 3,800 m up, find that rice takes forever to cook in an open pot. A steel needle laid gently on water floats, though steel is eight times denser than water. Honey poured on a cold morning creeps; warm, it runs. And an airplane weighing 70 tonnes climbs out of Kathmandu held up by nothing but air pushing a little harder under its wings than over them.

All of these are **fluid mechanics**: the physics of liquids and gases. A **fluid** is anything that flows: it has no shape of its own and takes the shape of its container, because its layers slide over one another under the smallest sideways force. The unit has two halves. **Fluid statics** studies fluids at rest: pressure, upthrust and floating (Archimedes), and the stretched "skin" of a liquid's surface (surface tension). **Fluid dynamics** studies fluids in motion: the internal friction called viscosity, the flow through a narrow tube, the fall of a small ball through oil, and the two great rules of steady flow, the equation of continuity and Bernoulli's equation.

## 3.1 Introduction

**Fluid statics** is the study of fluids at rest and the forces they exert. Two ideas carry most of it: **Archimedes' principle**, which explains why things float or sink, and **surface tension**, which explains why small things can rest on a water surface.

A STORY FROM SCIENCE

### Archimedes of Syracuse

**Archimedes** (about 287–212 BCE) lived in Syracuse, a Greek city on the island of Sicily. He worked out the areas and volumes of curved shapes by a method that anticipated calculus, found the law of the lever, and founded hydrostatics in his book *On Floating Bodies*, where he stated the principle that carries his name. He also designed machines: the screw for raising water that is still called an Archimedes screw, and engines that defended Syracuse against the Roman siege during which he was killed.

The famous story, told by the Roman architect Vitruvius two centuries later, is that King Hiero asked him whether a goldsmith had mixed silver into a gold crown. Stepping into a full bath and watching it overflow, he saw how to measure the crown's volume, and ran home shouting "Eureka!" ("I have found it!"). The story may be a legend, but the principle is real, and it is exactly how a jeweler can test gold today.

Some everyday observations this unit explains:

- Under water you feel the same pressure on your ears whichever way you turn your head.
- Rice is hard to cook in an open pot at Muktinath, and the air there is so dry that nosebleeds are common.
- At sea level the air presses on everything with about 101,325 Pa (1 atmosphere), yet we don't feel it; the pressure falls as you climb.
- Press on a liquid shut in a container and the pressure spreads equally everywhere in it: that is how a hydraulic lift and a car's brakes work.
- A bucket of water feels light while it is still under water, and suddenly heavy when it comes out.
- A water strider walks on a pond; a steel needle laid gently on water floats; bits of camphor dart about on water; a wet paintbrush's hairs cling together; a loop of thread on a soap film snaps into a circle when the film inside it is broken; water on a waxy leaf forms beads instead of spreading.



***Figure 3.1*** *Archimedes as imagined by the painter Domenico Fetti in 1620; no portrait from his own lifetime survives.*
Photo: Domenico Fetti, Public domain

### Pressure exerted by a fluid column

**Pressure** is the force acting normally (at right angles) on a surface, per unit area:

$$P = \frac{F}{A}$$

Its SI unit is the newton per square meter, N/m², called the **pascal** (Pa). A fluid at rest pushes at right angles on every surface it touches, so **fluid pressure** is the force the fluid exerts per unit area of any surface in it or around it.

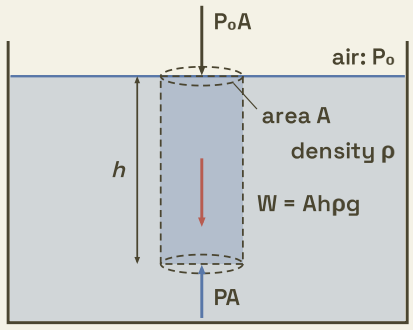



*Figure 3.2* *A column of liquid of height h, density ρ and cross-sectional area A. Its weight, spread over the area A at the bottom, gives the pressure hρg there, on top of the atmospheric pressure on its surface.*

Take a column of liquid of height $h$ and density ρ standing in a cylinder of cross-sectional area $A$ (Figure 3.2). The liquid's weight presses on the bottom.

1. The volume of the column is $V = Ah$, so its mass is $m = V\rho = Ah\rho$.
2. Its weight is $mg = Ah\rho g$, acting on the area $A$.
3. The pressure this weight makes at the bottom is

$$P = \frac{mg}{A} = \frac{Ah\rho g}{A} = h\rho g$$

The area cancels: the pressure at a depth $h$ doesn't depend on the shape or width of the vessel, only on the depth and the liquid. So the pressure due to a liquid column is

- **directly proportional to the depth**, $P \propto h$, and
- **directly proportional to the density of the liquid**, $P \propto \rho$.

The air above the surface pushes on it too, with the **atmospheric pressure** $P_0$, and a liquid passes that on downwards. So the **total (absolute) pressure** at a depth $h$ is

$$P = P_0 + h\rho g$$

The part $h\rho g$ alone, the pressure above atmospheric, is the **gauge pressure**: it is what a tire gauge or a blood-pressure cuff reads.

Some consequences:

- Water comes out faster from a lower tap on a tank than from a higher one: the pressure behind it is greater.
- Mercury is 13.6 times as dense as water, so 1 cm of mercury presses as hard as 13.6 cm of water.
- The air is a fluid too. Atmospheric pressure is the weight of the air above you, per square meter: greatest at sea level, less as you climb, because there is less air above you. In Bhaktapur, at about 1,400 m, it is about 86 kPa; at Muktinath, about 63 kPa.

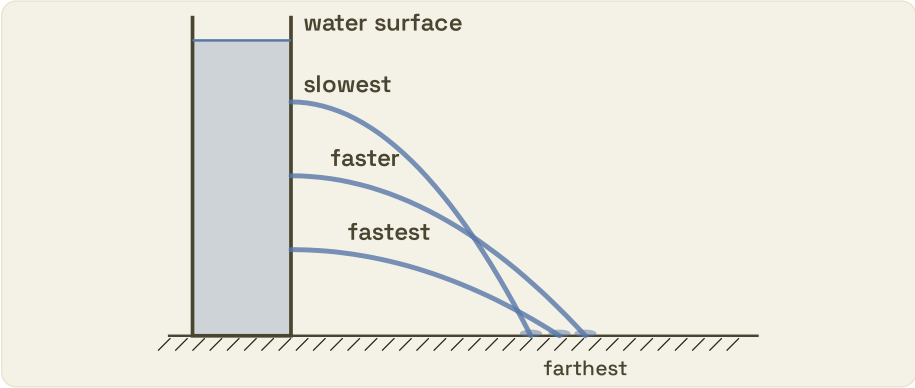



*Figure 3.3* *Water spurting from three holes in the side of a tank. The deeper the hole, the greater the pressure $h\rho g$ behind it, and the faster the water comes out. (How far each jet lands also depends on how far it has to fall, so the deepest jet isn't always the one that lands farthest.)*

DID YOU KNOW?

**Why rice won't cook at Muktinath**

Water boils when its vapor pressure reaches the pressure of the air above it. At Muktinath the air pressure is only about 63 kPa, so water boils at about 87 °C instead of 100 °C, and however hard you heat an open pot, it gets no hotter. Rice needs a higher temperature to soften quickly. A **pressure cooker** fixes this: its lid traps the steam, the pressure inside rises to about twice atmospheric, and the water boils at about 120 °C.

Nosebleeds at altitude have a different main cause: the cold, thin air holds very little water vapor, and breathing it dries and cracks the lining of the nose.

WORKED EXAMPLE

**Pressure at the bottom of Rara Lake**

Rara Lake in Mugu is about 167 m deep at its deepest, and the atmospheric pressure at its surface (about 2,990 m above sea level) is about 70 kPa. Find the gauge pressure and the total pressure at the deepest point. ($\rho = 1000$ kg/m³, $g = 9.8$ m/s²)

Answer

Gauge pressure: $h\rho g = 167 \times 1000 \times 9.8 = 1.64 \times 10^6$ Pa $=$ **1.64 MPa**.

Total pressure: $P = P_0 + h\rho g = 0.07 \times 10^6 + 1.64 \times 10^6 =$ **1.71 MPa**, about 17 times the atmospheric pressure at sea level. Every 10 m of water adds about one atmosphere.

Check 1

Three vessels with the same base area are filled with water to the same height: a narrow tube, a wide jar and a cone that widens upwards. The pressure on the base is:

a greatest in the cone, which holds the most water
b greatest in the narrow tube
c the same in all three
d greatest in the jar

## Pascal's law

Squeeze a water-filled polythene bag pricked with pinholes all over, and water spurts from every hole, not only from those opposite your hand. The push you give at one place reaches every part of the water.

**Pascal's law:** a pressure applied to an enclosed fluid (one shut in a container) is transmitted **undiminished** to every part of the fluid and to the walls of the container.

Two things follow. A fluid at rest presses equally **in all directions** at any point (you feel the same pressure on your ear however you turn your head). And the force it exerts on a surface is always **normal** to the surface, so the jets leave the bag (or the ball in Figure 3.4) at right angles to its skin.

The **hydraulic lift** uses this law to multiply a force. A small force $F_1$ on a small piston of area $A_1$ makes a pressure $F_1/A_1$, which reaches the large piston of area $A_2$ undiminished and pushes it with

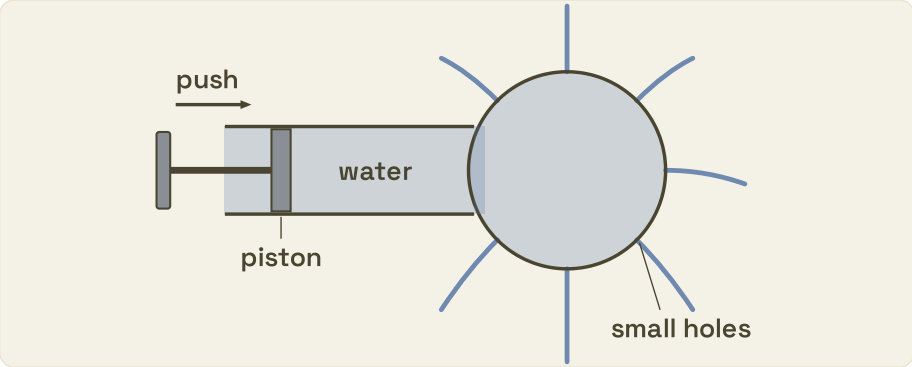



*Figure 3.4* *Pascal's syringe: a hollow glass ball with small holes all over it, on the end of a cylinder of water. Pushing the piston makes water spurt from every hole equally, each jet leaving at right angles to the ball's surface.*

$$F_2 = \frac{F_1}{A_1} \times A_2 = F_1 \frac{A_2}{A_1}$$

A piston 100 times larger in area pushes 100 times harder (moving 100 times less far, so no energy is created). Hydraulic jacks, car brakes, the arms of an excavator and the press that bales cotton all work this way.

### Specific gravity

The **specific gravity** (or **relative density**) of a substance compares its density with that of water:

$$\text{specific gravity} = \frac{\text{weight of a volume of the substance}}{\text{weight of the same volume of water at 4 °C}} = \frac{\text{density of the substance}}{\text{density of water at 4 °C}}$$

Water is densest at 4 °C, 1000 kg/m³ (1 g/cm³). So the specific gravity is just the density in g/cm³ without its unit: iron, 7.9; ice, 0.92; mercury, 13.6. Being a ratio of two quantities of the same kind, specific gravity has **no unit and no dimensions**.

The density of a liquid is measured with a **hydrometer**, a weighted glass float with a scale on its stem: it sinks deeper in a less dense liquid. A **lactometer** is a hydrometer made for milk: milk with water added is less dense, so the lactometer sinks deeper.

### Upthrust

Lift a bucket of water while it is still under water and it feels light; the moment it comes out it feels heavy. The water was pushing up on it. The upward force a fluid exerts on a body in it is the **upthrust** or **buoyant force**. Weigh a body in air and then in water: the **apparent loss of weight** is the upthrust.

$$\text{upthrust} = \text{weight in air} - \text{weight in the liquid}$$

Where does it come from? Pressure grows with depth.

1. Hold a cylinder of end area $A$ upright in a liquid of density $\sigma$, its top at depth $h_1$ and its bottom at depth $h_2$ (Figure 3.5).
2. The liquid presses down on the top with a force $P_1 A = (P_0 + h_1 \sigma g)A$, and up on the bottom with $P_2 A = (P_0 + h_2 \sigma g)A$. The pushes on the curved side are equal all round and cancel.
3. The net upward force is

$$U = P_2 A - P_1 A = (h_2 - h_1)\sigma g A = lA\sigma g = V\sigma g$$

where $V = lA$ is the volume of the cylinder, which is the volume of liquid it pushes aside (**displaces**). And $V\sigma g$ is the **weight of the displaced liquid**.

Any shape can be built from thin upright cylinders, so the result holds for every body:





27. Water flows from a narrow pipe into a wider one.
    a. State Bernoulli's theorem.
    b. Give two applications of Bernoulli's principle in real life.
    c. Water flows at 0.5 m/s through a horizontal pipe of radius 0.02 m and length 1 m. The viscosity of water is 0.001 N s/m², and the pressure difference across the pipe is 500 Pa. Calculate the power lost to viscosity along the pipe.

28.
    a. To what kind of fluid does Bernoulli's principle apply?
    b. Derive Bernoulli's equation for the streamline flow of an incompressible, non-viscous fluid.
    c. Water flows through a horizontal pipe. At one point its speed is 2 m/s and its pressure is 3000 Pa above atmospheric. At another point its speed has risen to 4 m/s. Find the pressure at the second point, assuming steady flow.

29. A horizontal pipe has a varying cross-section, and water flows steadily through it.
    a. State and derive the equation of continuity for incompressible flow.
    b. State Bernoulli's principle and write its mathematical form.
    c. Why is the pressure of a fluid lower where its speed is higher?
    d. The pipe tapers from a diameter of 10 cm to 5 cm. Water flows at 2 m/s in the wider part. Find i. its speed in the narrow part, ii. the pressure difference between the two ends. ($\rho = 1000$ kg/m³)

A companion to your textbook, not a replacement for it: study from the book your school uses, and where the two differ, follow your teacher. Found a mistake? Please tell us.

18. A clean glass capillary tube is dipped vertically into water.
    a. Define surface tension. What is its SI unit?
    b. What is capillarity? Derive the expression for the capillary rise.
    c. A capillary tube of radius 0.25 mm is dipped in water. If the surface tension is 0.072 N/m and the angle of contact zero, calculate the height to which the water rises. ($g = 9.8$ m/s², $\rho = 1000$ kg/m³)

19.
    a. What is the angle of contact? Explain how it affects capillary action.
    b. Derive the relation between surface tension, capillary rise and the radius of the tube.
    c. Two small drops of mercury, each of radius $r$, join to form a single larger drop. Calculate the ratio of the total surface energies before and after.

20. Oil flows through a narrow tube under pressure.
    a. Define viscosity and give its SI unit.
    b. State and explain Newton's law of viscosity.
    c. A fluid of viscosity 0.01 N s/m² flows through a tube of radius 2 mm. If the velocity gradient is 400 $s^{-1}$, find the shear stress.

21.
    a. Explain why small drops of liquid are spherical.
    b. Discuss how adhesive and cohesive forces affect the shape of drops.

22.
    a. i. Define surface energy.
    ii. Derive an expression for surface energy in terms of surface tension and area.
    b. Calculate the work done in breaking a drop of water 2 mm across into a million droplets of the same size. (Surface tension of water = $72 \times 10^{-3}$ N/m)

23.
    a. Define viscosity and the coefficient of viscosity.
    b. Derive an expression for the force needed to move one layer of fluid over another.
    c. A plate of area 0.2 m² moves at 0.1 m/s over a thin layer of liquid 0.5 mm thick. If the viscosity of the liquid is 0.5 N s/m², find the force needed to keep the plate moving.

24.
    a. State and explain Stokes' law.
    b. Derive an expression for the terminal velocity of a sphere falling through a viscous fluid.
    c. A ball of radius 0.002 m falls through oil of viscosity 0.1 N s/m². If the density of the ball is 900 kg/m³ and that of the oil 700 kg/m³, calculate its terminal velocity. ($g = 9.8$ m/s²)

25.
    a. State Poiseuille's formula for the flow of a liquid through a capillary tube.
    b. Explain the importance of Poiseuille's equation in fluid dynamics.
    c. A liquid of viscosity 0.6 N s/m² flows through a tube of radius 2 mm. If the pressure difference across 10 cm of the tube is 500 Pa, calculate the volume flow rate.

26.
    a. What causes surface tension? Why does hot soup taste better than cold soup?
    b. In the figure, water flows through a horizontal pipe with $A_1 = 0.02$ m², $A_2 = 0.01$ m², $P_1 = 4 \times 10^4$ Pa and $v_1 = 2$ m/s. Calculate i. the speed at $A_2$, ii. the pressure at $A_2$.

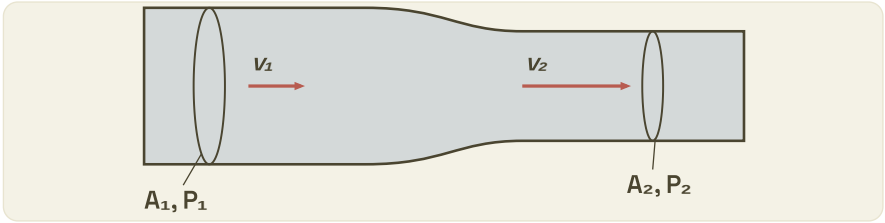



*Figure 3.35* *For question 26b: water flowing through a horizontal pipe from the wide end, area $A_1$, to the narrow end, area $A_2$.*





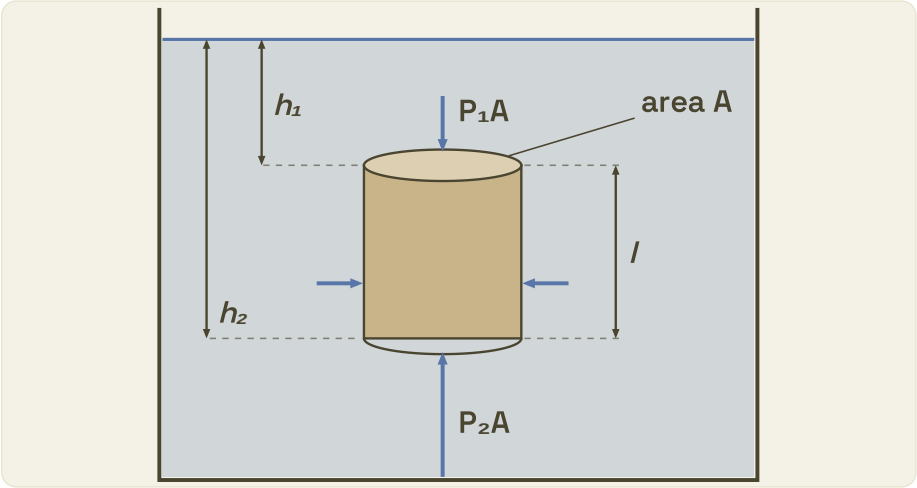



*Figure 3.5* *A cylinder of height $l$ and end area $A$ held under water, its top at depth $h_1$ and its bottom at $h_2$. The liquid pushes down on the top with $P_1A$ and up on the bottom with the larger $P_2A$; the sideways pushes cancel.*

$$U = V\sigma g = \text{weight of the fluid displaced}$$

The upthrust depends only on the volume of fluid displaced (the volume of the part of the body that is under the surface) and on the density of the fluid. It doesn't depend on what the body is made of, whether it is solid or hollow, or how deep it is (once it is fully under). A denser liquid gives more upthrust: an iron ball sinks in water but floats in mercury.

### Archimedes' principle

**Archimedes' principle:** when a body is wholly or partly immersed in a fluid, it experiences an upthrust equal to the weight of the fluid it displaces.

It holds in gases too: a hydrogen balloon rises because the weight of the air it displaces is more than its own weight.

### Floating and sinking

A body in a liquid has two forces on it: its **weight** $W$, down, and the **upthrust** $U$, up.

- If $W > U$ when it is fully under, it **sinks**. For a solid body this happens when its density is greater than the liquid's.
- If $W < U$ when it is fully under, it rises, breaks the surface and comes to rest with only part of it under: it **floats**.
- If $W = U$ when it is fully under, it stays wherever it is put, neither rising nor sinking (a fish or a submarine does this by adjusting its density).

**Law of flotation:** a floating body displaces a weight of liquid equal to its own weight. If a body of volume $V$ and density $\rho$ floats in a liquid of density $\sigma$ with a volume $v$ under the surface, then $V\rho g = v\sigma g$, so

$$\frac{v}{V} = \frac{\rho}{\sigma}$$

**The fraction of a floating body that is under the surface equals the ratio of its density to the liquid's.** Wood of density 600 kg/m³ floats in water with 60 % of it under; ice (920 kg/m³) floats in sea water (1030 kg/m³) with $920/1030 = 89\%$ under, which is where "the tip of the iceberg" comes from.

WORKED EXAMPLE

**A plank on the Bagmati**

A wooden plank of volume 0.05 m³ floats on water with 70 % of its volume under the surface. Find the plank's density and its mass. ($\rho$ of water = 1000 kg/m³)

Answer

By the law of flotation, density of plank = fraction under × density of water = $0.70 \times 1000 =$ **700 kg/m³**.
Mass = density × volume = $700 \times 0.05 =$ **35 kg**.

**Check 2**

A boat floating on a pond is carrying a heavy stone. The stone is thrown overboard and sinks. The water level of the pond:

a rises
b falls
c stays the same
d rises, then falls back

### Equilibrium of a floating body

A floating body is balanced by two equal forces along one vertical line, but it can still tip over. Whether it does depends on two points (Figure 3.7):

- the **center of gravity** G, through which its weight acts downwards;
- the **center of buoyancy** B, the center of gravity of the liquid it displaces, through which the upthrust acts upwards.

For a body floating at rest, $W = U$, and G and B lie on the same vertical line.

When the body tilts, the shape of the liquid it displaces changes: more of one side goes under. So the center of buoyancy moves, to a new point B′ on the side that went deeper; where exactly depends on the new shape of the displaced liquid. The upthrust now acts up through B′, and the weight still down through G, so they no longer act along one line: they form a **couple** that turns the body.

The point where the vertical line through the new center of buoyancy B′ meets the body's center line (the line through G and B when it was upright) is the **metacenter** M. The distance GM is the **metacentric height**.

- **M above G: stable equilibrium.** The couple turns the body back upright (Figure 3.7 b).
- **M below G: unstable equilibrium.** The couple turns it further over, and it capsizes (Figure 3.7 c).
- **M at G: neutral equilibrium.** There is no couple, and the body stays tilted at its new angle.

It is the metacenter, not the center of buoyancy, that has to be above G: in a stable ship B is usually **below** G.

This is why the bottom of a ship is made heavy, with heavy engines and ballast placed low: lowering G keeps it well below M. It is also why passengers in a small boat on Phewa Lake are asked to sit down rather than stand: standing raises G. A round log can roll over at the slightest push because its metacenter is at its center and so, for a uniform log, at its center of gravity: it is in neutral equilibrium.





**Check 21**

10. Water flows at 2 m/s from a hole 1 mm² in area in the bottom of a beaker. How high is the water in the beaker at that moment? (Torricelli: $v = \sqrt{2gh}$; g = 10 m/s²)

a 20 cm
b 10 cm
c 12 cm
d 15 cm

**Check 22**

11. According to Poiseuille's law, which changes the volume of liquid flowing through a pipe most?

a The pressure difference
b The length of the pipe
c The radius of the pipe
d The viscosity of the liquid

**Check 23**

12. Which best explains the lift on an aircraft's wing?

a The pressure is higher above the wing
b Air moves faster above the wing
c Air moves slower above the wing
d Gravity pulls the wing upwards

14.
a. Why is the bottom of a ship made heavy?
b. Does Archimedes' principle apply in a satellite moving in a circular orbit round the Earth? Explain.
c. A string holds a solid iron object of mass 200 g totally under a liquid of density 900 kg/m³. If the density of iron is 8000 kg/m³, calculate the tension in the string. (g = 10 m/s²)

15. A wooden block floats in water, partly under it.
a. State Archimedes' principle.
b. Explain the conditions for a body to float or sink in a liquid.
c. A block of volume 0.02 m³ floats in water with 75 % of its volume under the surface. Find its density.

16.
a. How does Archimedes' principle apply in an artificial satellite?
b. The specific gravity of sea water is 1.03 and that of ice 0.92. What fraction of an iceberg is above the surface of the water?
c. A block of ice 25 cm thick floating on fresh water can just support an 80 kg man standing on it. What is the smallest area the block can have? (Specific gravity of ice = 0.917)

17.
a. Define surface energy.
b. Derive the relation between surface energy and surface tension.
c. A rectangular plate 6 cm × 4 cm × 2 mm rests with its largest face on the surface of water. Calculate the downward force on it due to surface tension, taking the angle of contact as zero. Then find the downward force if the plate is held vertically with only its longest edge just touching the water. (Surface tension of water = $7 \times 10^{-2}$ N/m)

## Exercise

### Choose the correct option

Check 12

1. Which of these liquids is likely to have the highest surface tension?

a Alcohol
b Water
c Mercury
d Oil

Check 13

2. Which of these is not caused by surface tension?

a Capillary action
b A small insect floating on water
c The expansion of a liquid's volume on heating
d The formation of drops

Check 14

3. Which part of a liquid shows surface tension?

a Its free surface
b Its middle
c Where it touches the wall
d Near the bottom

Check 15

4. A liquid rises to a height $h$ in a capillary tube on Earth. In the same tube on the Moon's surface (where $g$ is a sixth of Earth's), it would rise to:

a $h$
b $h/6$
c $6h$
d $1.6h$

Check 16

5. A liquid rises 7.5 cm in a capillary of radius 2 mm. How high does it rise in a capillary of radius 1 mm?

a 7.5 cm
b 15 cm
c 10 cm
d 5 cm

Check 17

6. A body floats on water in a beaker, and the whole system is falling freely under gravity. The upthrust on the body is:

a equal to the weight of the water displaced
b equal to the weight of the body
c equal to the loss in weight of the body
d zero

Check 18

7. A tank holds water 5 m deep above a hole 1 $cm^2$ in area in its bottom. The water first flows out at a rate of about ($g = 10\ m/s^2$):

a $10^{-3}\ m^3/s$
b $10^{-4}\ m^3/s$
c $10\ m^3/s$
d $10^{-2}\ m^3/s$

Check 19

8. Two solids float in water, one with $\frac{2}{3}$ of its volume under water and the other with $\frac{1}{2}$. The ratio of their densities is:

a 1 : 2
b 2 : 3
c 3 : 4
d 4 : 3

Check 20

9. Water flows at 4 m/s through a horizontal pipe. Where the pipe's diameter doubles, its speed is:

a 1 m/s
b 2 m/s
c 4 m/s
d 8 m/s

### Answer these questions





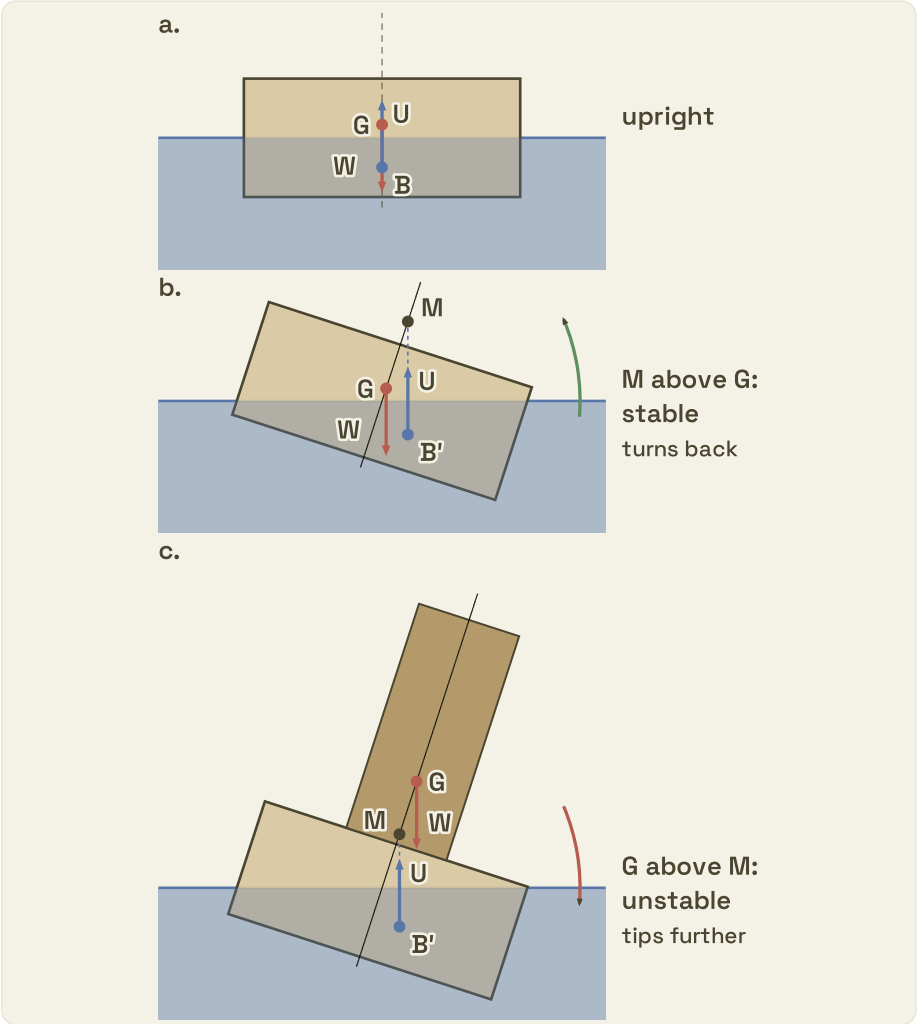



*Figure 3.7* *a. A floating boat, upright: its weight W acts down through G, the upthrust U up through B, the center of the water it displaces. b. Tilted, the displaced water's shape changes and B moves to B′, towards the side that went deeper. The line of the upthrust through B′ meets the boat's center line at the metacenter M, above G: the two forces make a couple that turns the boat back upright. c. With G high (a top-heavy load), M is below G, and the same couple turns the boat further over.*

CAREFUL

**The metacenter decides, not the center of buoyancy**

Some books say a floating body is stable if "the center of buoyancy is above the center of gravity". In most ships it isn't: B is below G. A floating body is stable when the **metacenter** is above the center of gravity, which keeps G below M.

## 3.2 Surface tension

A STORY FROM SCIENCE

**Thomas Young**

**Thomas Young** (1773–1829), an English doctor and physicist, is best known for showing that light is a wave (Unit 7). In 1805 he published an essay on the cohesion of fluids that explained surface tension and the angle at which a liquid meets a solid, the **angle of contact** (Section 3.4); the French mathematician Pierre-Simon Laplace worked out the same ideas independently the next year. Young also deciphered part of the Rosetta Stone and gave his name to Young's modulus (Grade 11, Unit 8).

### The surface behaves like a stretched skin

A water strider stands on a pond, its feet making small dimples in the surface as if it were standing on a trampoline. A drop of water hanging from a tap is pulled into a rounded bag. Small drops of rain and dew are nearly perfect spheres.

The free surface of a liquid behaves like a **stretched elastic membrane**, always trying to shrink to the **smallest possible area**. (Picture someone curling up into a ball to make as little of themselves as possible exposed.) This property is **surface tension**.



*Figure 3.8* *Thomas Young, engraved after a portrait by Thomas Lawrence.*
Photo: Wikimedia Commons, Public domain

To give it a number, imagine a line of length $L$ drawn on the surface. The surface on one side pulls on the surface on the other side, along the surface and at right angles to the line. The **surface tension** $T$ of a liquid is this **force per unit length** of a line drawn in its surface:



*Figure 3.9* *A water strider stands on a stream's surface. Its feet dimple the surface without breaking it, and the stretched surface holds it up.*
Photo: Phyzome is Tim McCormack, CC BY-SA 3.0

$$T = \frac{F}{L}$$

Its SI unit is the newton per meter, N/m (equivalently J/m², Section 3.3), and its dimensional formula is $\frac{[\mathrm{MLT}^{-2}]}{[\mathrm{L}]} = [\mathrm{ML}^{0}\mathrm{T}^{-2}]$.

| Liquid (at 20 °C, in air) | Surface tension (N/m) |
|---|---|
| Mercury | 0.485 |
| Water | 0.0728 |
| Olive oil | 0.032 |
| Ethanol (alcohol) | 0.022 |
| Soap solution | about 0.025 |

Surface tension is a property of a liquid's surface where it meets air or another substance it doesn't mix with. A gas has no free surface of its own (it spreads to fill its container), so a gas has no surface tension. It lets a body denser than the liquid, like a needle, rest on the surface, as long as the body doesn't break through.





YOUR TURN

**Two sheets of paper**

Hold two sheets of paper hanging down, a few centimeters apart, and blow hard between them. Do they fly apart or come together? Then hold one sheet just under your lower lip, so that it droops, and blow straight out over its top.

What you should see

The sheets come **together**, and the drooping sheet **rises**. The fast air you blow has a lower pressure than the still air on the other sides, which pushes the paper into the stream. (Strictly, the air coming out of your mouth is at room pressure; the paper's curve and the stream's pull on the air near it lower the pressure beside the stream. But the result is the same lesson: a fast stream has a lower pressure beside it than still air.)

IN SHORT

**Key ideas of Unit 3**

- Pressure in a liquid at depth $h$: $P = P_0 + h\rho g$ (gauge pressure $h\rho g$), the same in all directions, whatever the vessel's shape. Pascal's law: pressure applied to an enclosed fluid is passed on undiminished; a hydraulic lift gives $F_2 = F_1 \frac{A_2}{A_1}$.
- Archimedes' principle: upthrust = weight of the fluid displaced. A body floats if its average density is less than the liquid's; floating, $\frac{v}{V} = \frac{\rho}{\sigma}$. A floating body is stable when its **metacenter** is above its center of gravity.
- Surface tension $T = F/L$ (N/m, $[\mathrm{ML}^{0}\mathrm{T}^{-2}]$), from the unbalanced pull on the surface molecules; it falls as the temperature rises and with most impurities. Surface energy: $W = T\,\Delta A$.
- Angle of contact: under 90° (water on glass, the liquid wets it and rises), over 90° (mercury, it falls). Capillary rise: $h = \frac{2T\cos\theta}{r\rho g}$, so $hr$ is constant.
- Viscosity: internal friction between layers. Newton: $F = -\eta A \frac{dv}{dx}$; $\eta$ in N s/m² = Pa s (1 poise = 0.1 Pa s), $[\mathrm{ML}^{-1}\mathrm{T}^{-1}]$. $\eta$ of a liquid falls as it warms; of a gas, it rises.
- Laminar flow below the critical velocity, turbulent above it; Reynolds number $R_e = \frac{\rho v D}{\eta}$, laminar below about 2000 in a pipe.
- Poiseuille: $V = \frac{\pi P r^4}{8\eta l}$. Stokes: $F = 6\pi\eta r v$; terminal velocity $v = \frac{2r^2(\rho - \sigma)g}{9\eta}$.
- Continuity: $A_1 v_1 = A_2 v_2$. Bernoulli: $P + \frac{1}{2}\rho v^2 + \rho g h =$ constant; where a fluid moves faster, its pressure is lower. Torricelli: $v = \sqrt{2gh}$.

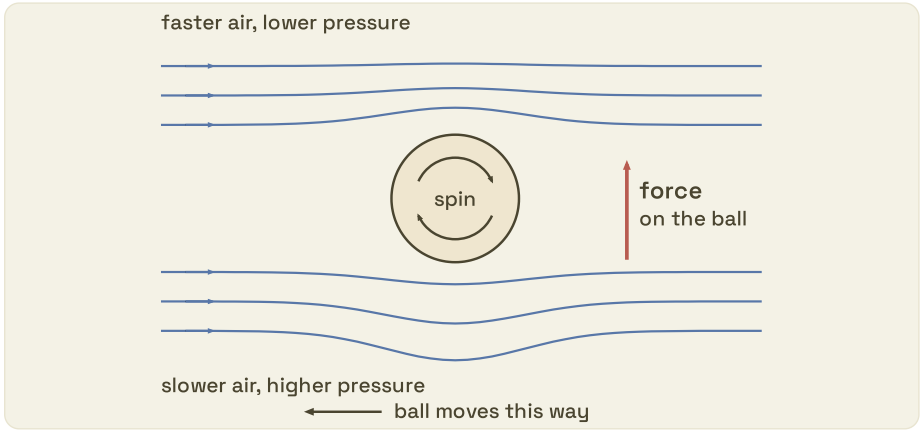



*Figure 3.33* *A ball spinning clockwise, seen from above, with the air streaming past it from left to right (the ball moving to the left through still air). On top, the ball's surface moves with the airstream: the air is faster, the pressure lower. Below, it moves against it: slower air, higher pressure. The ball is pushed towards the low-pressure side.*

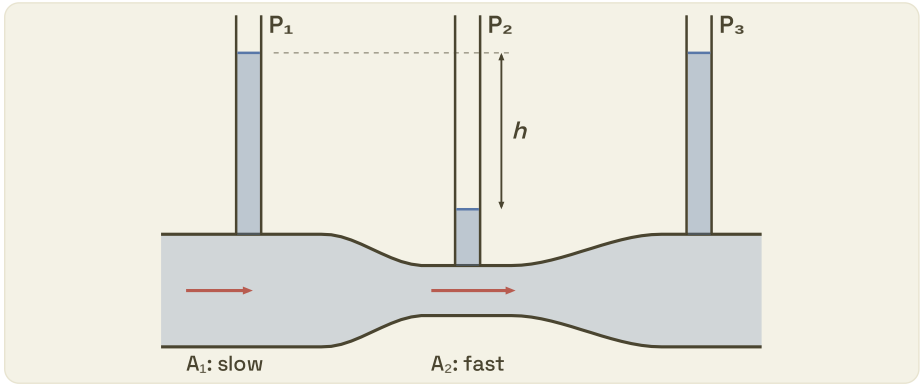



*Figure 3.34* *A venturimeter. Water flows from the wide part (area $A_1$) into the narrow throat (area $A_2$) and out into the wide pipe again. Upright tubes show the pressure: the water stands high where the flow is slow ($P_1$, $P_3$) and low at the throat, where it is fastest ($P_2$).*

$$v_1 = A_2\sqrt{\frac{2gh}{A_1^2 - A_2^2}}, \qquad \text{rate of flow } V = A_1 v_1 = A_1 A_2\sqrt{\frac{2gh}{A_1^2 - A_2^2}}$$

Medical devices use the same effect: a **Venturi mask** gives a patient oxygen mixed with air in an exact proportion, the fast oxygen jet drawing in room air through side openings, and a **nebulizer** turns a medicine into a mist to breathe in, just as an atomizer does.

More everyday Bernoulli:

- **Roofs in a storm.** Fast wind over a roof lowers the pressure above it while the air inside stays at the ordinary pressure: light tin roofs are lifted off in a storm, which is why they are tied down.
- **Standing near a train or a fast bus.** The air between you and the fast vehicle is dragged along fast, its pressure drops, and you feel pulled towards it: stand back from the platform's edge.
- **A chimney draws better in wind:** wind over the top lowers the pressure there.





**Surface tension decreases as the temperature rises**, and it is lowered by many **impurities** dissolved in the liquid, such as soap and detergents. (Some dissolved substances, like common salt, raise it slightly.) This explains:

- **Antiseptic lotions** have low surface tension, so they spread easily into every part of a cut or wound.
- **Soap and detergents** lower the surface tension of water, so it spreads into the fibers of cloth and under grease instead of sitting in drops on it.
- **Camphor "dances" on water.** As camphor dissolves, it lowers the surface tension around it, more on one side than the other where the piece is uneven; the stronger pull of the clean water on the other side drags it along.
- **Hot soup tastes better than cold soup:** its surface tension is lower, so it spreads over more of the tongue (and more of its flavors evaporate up into the nose).

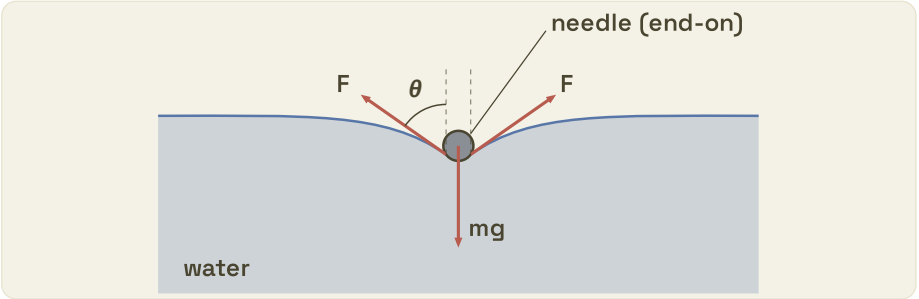



*Figure 3.10* *A needle seen end-on, resting in a dip in the water surface. The surface pulls along itself on each side with a force F (per unit length T), at an angle θ to the vertical. Their vertical parts add up and hold up the weight mg; their sideways parts cancel.*

The needle presses a shallow groove into the surface (Figure 3.10). Along each side of the groove, the surface pulls on the needle along itself, a force $F = TL$ for a needle of length $L$, tilted at an angle θ to the vertical. The sideways parts of the two pulls cancel; the vertical parts add up to $2TL\cos\theta$, which holds up the needle's weight:

$$2TL\cos\theta = mg$$

Soap lowers $T$ so much that the surface can no longer hold the weight, and the needle sinks.

More examples of surface tension:

- **A wet paintbrush.** Under water, a brush's hairs spread out freely. Lift it out and they cling together in a point: now there is a water surface around the hairs, and by pulling the hairs together the water's surface makes its area smaller.
- **A loop of thread on a soap film.** Lay a loop of thread gently on a soap film stretched across a wire ring: the loop lies in any odd shape. Break the film inside the loop with a hot pin and the loop springs into a **perfect circle** (Figure 3.11). The film outside pulls the thread outwards equally all round to make its own area as small as possible, which means making the hole as large as possible, and of all shapes with a given perimeter the circle encloses the largest area.

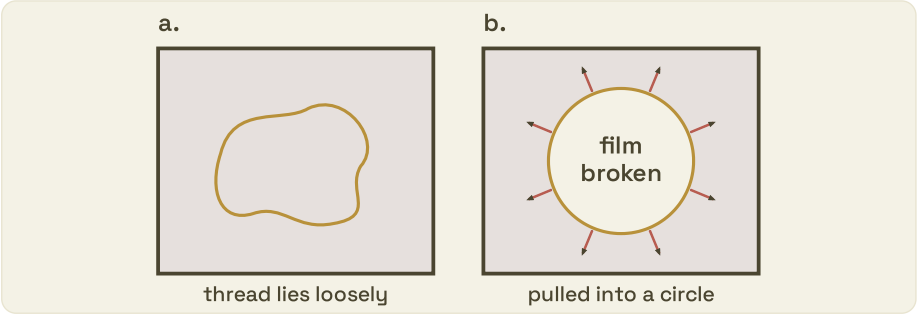



*Figure 3.11* *a. A loop of thread lies loosely on a soap film: the film pulls on it equally from both sides. b. The film inside the loop is broken: now the film outside pulls the thread outwards all round, with the same force T per unit length, and the loop becomes a circle.*

YOUR TURN

**Holding up a needle**

A steel needle 4.0 cm long with a mass of 0.12 g rests on water (T = 0.072 N/m). At what angle θ to the vertical does the surface pull on it? (g = 9.8 m/s²)

Answer

$2TL\cos\theta = mg$, so $\cos\theta = \dfrac{mg}{2TL} = \dfrac{0.12\times10^{-3}\times9.8}{2\times0.072\times0.040} = \dfrac{1.18\times10^{-3}}{5.76\times10^{-3}} = 0.204$, so θ ≈ **78°**. The groove is shallow: the surface pulls almost sideways. A needle twice as heavy would need $\cos\theta = 0.41$, θ ≈ 66°; one more than five times as heavy ($\cos\theta > 1$) couldn't be held up at all.

### Molecular theory of surface tension

The molecules of a liquid attract each other. The attraction between molecules of the same substance is **cohesion**. It acts only over a tiny distance, about $10^{-9}$ m (a few molecules' widths), called the **range of molecular attraction**; imagine a small sphere of that radius round each molecule, its **sphere of influence**.

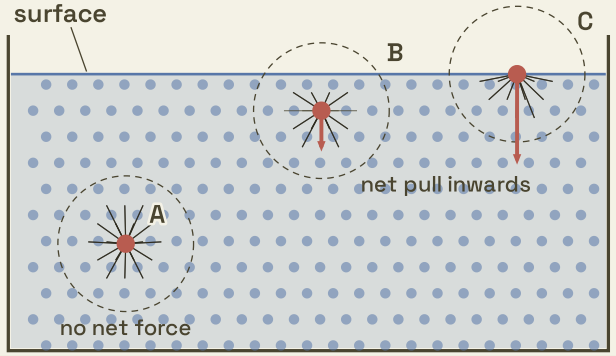



*Figure 3.12* Three molecules and their spheres of influence. A, deep inside, is pulled equally in all directions: no net force. B, just under the surface, has fewer molecules above it than below: a small net pull downwards. C, on the surface, has liquid only below it: the largest net pull into the liquid.

- A molecule **deep inside** the liquid (A in Figure 3.12) is surrounded by molecules on all sides. They pull it equally in every direction: the net force is zero.
- A molecule **near the surface** (B), less than a molecular range from it, has part of its sphere of influence sticking out into the air, where there are hardly any molecules. It is pulled down more than up: a net force into the liquid.
- A molecule **on the surface** (C) has liquid only below it. It feels the largest net pull inwards.

So every molecule in the surface layer is pulled into the liquid. To bring a molecule from inside up to the surface, work must be done against this pull: surface molecules have **more potential energy** than those inside. A system settles where its potential energy is least, so the liquid keeps as few molecules as possible in its surface: it makes its **surface area as small as it can**. The surface behaves as though it were under tension, like a stretched membrane.

That is why a small drop is a **sphere**: of all shapes holding a given volume, a sphere has the smallest surface area. (A large drop is flattened by its weight, which matters more for large drops; in space, where things are weightless, even a large blob of water floats as a sphere.)

Check 3

A molecule in the surface of a liquid has, compared with a molecule deep inside:

a less potential energy, because it is pulled less

b more potential energy, because work was done to bring it there against an inward pull

c the same potential energy, since the net force on both is zero

d no forces on it at all





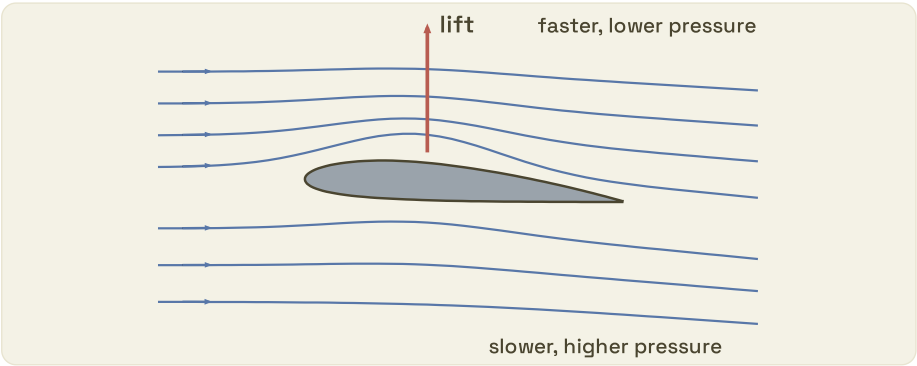



*Figure 3.31* Air flowing past an aircraft wing tilted slightly up into the airstream. The streamlines crowd together over the curved top, where the air moves faster and the pressure is lower; below the wing the air is slower and the pressure higher. The air leaving the wing is turned downwards. The pressure difference gives the lift.

CAREFUL

**Not "equal transit time"**

Some books explain lift by saying that the air over the top must travel farther and so must go faster to "meet up" with the air below at the trailing edge. That isn't so: the air over the top actually arrives at the trailing edge well **before** the air below, and nothing makes the two meet. The reliable statement is that the wing's shape and tilt make the air above move faster, so the pressure there is lower; the same wing also turns the air passing it downwards, and by Newton's third law the air pushes the wing up. Both descriptions are true at once. A flat, tilted wing, or an airplane flying upside down, still gets lift from its tilt.

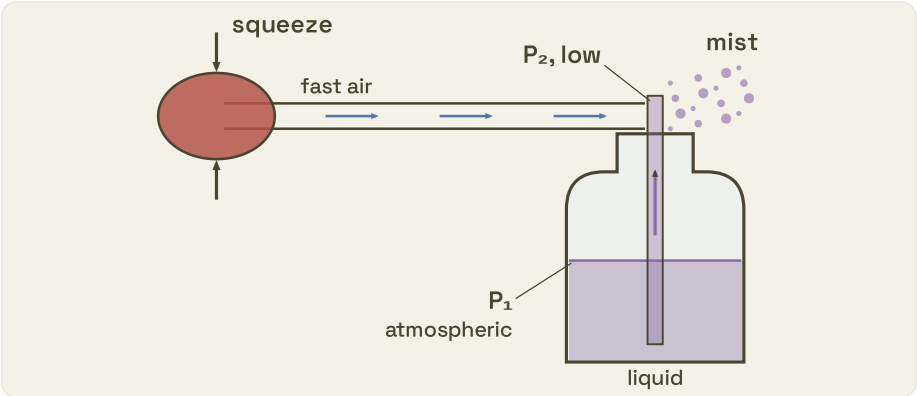



*Figure 3.32* An atomizer. Squeezing the bulb (or pushing the piston) blows air fast across the top of a narrow tube dipping into the liquid. The pressure there, $P_2$, falls below the pressure $P_1$ on the liquid's surface in the container, so the liquid is pushed up the tube and sprayed out as a mist.

clings to the ball longer before breaking away. The air leaving the ball is thrown towards the other side, so the ball swings towards the side the seam points to.)

**5. The Venturi effect and the venturimeter.** Where a pipe narrows (a **venturi**), the fluid speeds up and its pressure drops. A **venturimeter** measures the rate of flow in a pipe from that pressure drop: vertical tubes (or a pressure gauge) at the wide part and at the throat show the difference (Figure 3.34).

If the water levels in the two tubes differ by $h$, the pressure difference is $P_1 - P_2 = \rho g h$. Continuity gives $v_2 = \dfrac{A_1}{A_2}v_1$, and Bernoulli for a horizontal pipe gives $P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_1^2)$. Putting them together,

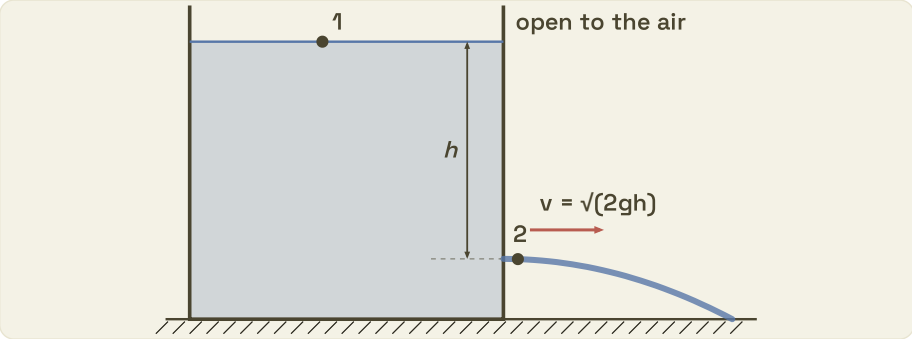



*Figure 3.30* *A wide tank, open at the top, with a small hole a depth h below the water surface. The water spurts out at $v = \sqrt{2gh}$, as fast as if it had fallen freely from the surface.*

2. The tank is wide and the hole small, so the surface sinks very slowly: $v_1 \approx 0$.
3. Take the hole as the reference level: $h_1 = h$, $h_2 = 0$.

$$P_0 + 0 + \rho g h = P_0 + \tfrac{1}{2}\rho v^2 + 0 \quad \Rightarrow \quad v = \sqrt{2gh}$$

**Torricelli's theorem** (Evangelista Torricelli, 1643, a century before Bernoulli): the speed of efflux from a hole a depth $h$ below the surface is the speed a body gains falling freely through $h$. The volume per second coming out of a hole of area $a$ is $av = a\sqrt{2gh}$. (A real jet comes out a little slower, and narrows just past the hole, because of viscosity.) The tank must be **open** at the top (or the air above the water at atmospheric pressure): in a sealed bottle, the air above the water thins as water leaves, and the flow soon stops, which is why a sealed oil tin needs a second hole to pour well.

YOUR TURN

**Water from a tank**

A large open water tank on a roof has a tap 3.2 m below the water surface. How fast does water come out of the tap when it is opened fully? If the tap's opening is 1.5 cm² in area, how long does it take to fill a 20-liter bucket? (g = 9.8 m/s²; ignore friction in the pipe)

Answer

$v = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 3.2} = \sqrt{62.7} = \mathbf{7.9\ m/s}$.
Flow = $av = 1.5 \times 10^{-4} \times 7.92 = 1.19 \times 10^{-3}$ m³/s = 1.19 L/s. Time for 20 L = 20 ÷ 1.19 ≈ **17 s**. (In a real house the pipes' friction makes it slower.)

### Applications of Bernoulli's principle

**1. The lift on an airplane's wing.** A wing (an **airfoil**, or aerofoil) is curved and tilted slightly upwards to the oncoming air. The air flowing over its top surface is made to follow the curve and moves faster than the air below. By Bernoulli's principle, the pressure above the wing is lower than below it. The pressure difference, acting on the large area of the wing, gives an upward force, the **lift**, which lets the aircraft take off, climb and stay up.

**2. Helicopters.** A helicopter's rotor blades are wings that go round instead of forwards. Each blade, cutting through the air, has faster air and lower pressure above it, which lifts the helicopter, and it turns the air downwards in a strong downdraft. Because it needs no forward speed to make lift, a helicopter can hover, rise straight up and land on a small pad, which is why helicopters carry out rescues from high in the Himalaya.

**3. Sprayers and atomizers.** In a perfume spray, a paint sprayer or the old carburetor of a motorbike, a jet of air is blown fast across the top of a narrow tube whose bottom dips into the liquid. The fast air has low pressure, so the atmospheric pressure on the liquid in the container pushes it up the tube; at the top the air jet breaks it into a fine mist of droplets.

**4. Swing of a spinning ball: the Magnus effect.** A spinning ball drags a thin layer of air round with it. On the side where the ball's surface moves the same way as the air streaming past it, the air is helped along and moves faster; on the other side, where the surface moves against the air, the air is slowed. The pressure is lower on the faster side, and the ball is pushed sideways towards it: its path curves. This is the **Magnus effect**. It makes a football kicked with sidespin bend round a wall of defenders, a table-tennis ball with topspin dip sharply, and a baseball pitcher's curveball curve. (The swing of a cricket ball is a different effect. The bowler keeps the seam upright but angled to one side; the seam makes the air on that side turbulent, and turbulent air



## 3.3 Surface energy

To make a liquid's surface larger, molecules must be brought from inside to the surface against the inward pull, and that takes work. The extra potential energy of the molecules in the surface is the **surface energy**. The **surface energy per unit area** is the work needed to increase the surface area by one unit (at constant temperature). A liquid always tends to lower its surface energy by making its surface area smaller, which is why drops form.

### Relation between surface energy and surface tension

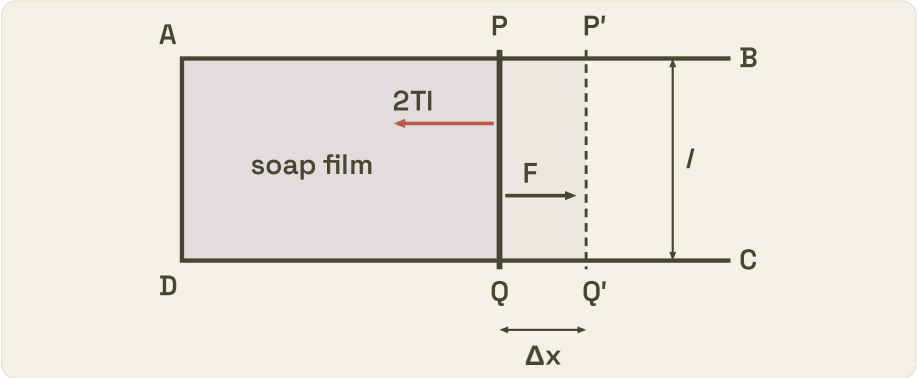



*Figure 3.13* *A soap film on a wire frame ABCD with a sliding wire PQ of length l. The film's two surfaces pull PQ inwards with a force 2Tl. Pulling PQ out by Δx to P′Q′ adds an area l Δx to each surface.*

1. Take a rectangular wire frame with one side, PQ, a wire that can slide (Figure 3.13). Dip it in soap solution, so that a film forms on it. A film has **two surfaces**, front and back, each pulling on PQ with a force $T$ per unit length.
2. If PQ has length $l$, the total inward force on it is
$$F = 2Tl$$
(Left alone, PQ slides in and the film shrinks.)
3. Pull PQ outwards slowly, with a force just equal to $F$, through a small distance Δx, to P′Q′. The work done is
$$W = F\,\Delta x = 2Tl\,\Delta x$$
4. Each surface has grown by an area $l\,\Delta x$, so the total increase in area is
$$\Delta A = 2l\,\Delta x$$
5. Dividing, $W = T \times 2l\,\Delta x$, so
$$W = T\,\Delta A \qquad \text{and} \qquad T = \frac{W}{\Delta A}$$

The work done is stored as surface energy (when the temperature is kept constant). So **the surface energy per unit area is numerically equal to the surface tension**, and the surface energy of an area ΔA is $T\,\Delta A$. Surface tension can be written in J/m² as well as N/m: the two units are the same, since 1 J/m² = 1 N m/m² = 1 N/m.

WORKED EXAMPLE

**Eight drops become one**

Eight identical water drops, each of radius 1.0 mm, join into one big drop. Find the energy released. (T = 0.072 N/m)

Answer

The volume is unchanged: $\frac{4}{3}\pi R^3 = 8 \times \frac{4}{3}\pi r^3$, so $R = 2r = 2.0$ mm.
Area before: $8 \times 4\pi r^2 = 32\pi r^2$. Area after: $4\pi R^2 = 16\pi r^2$.
The area falls by $\Delta A = 16\pi r^2 = 16\pi \times (1.0 \times 10^{-3})^2 = 5.03 \times 10^{-5}$ m².
Energy released = $T\,\Delta A = 0.072 \times 5.03 \times 10^{-5} = \mathbf{3.6 \times 10^{-6}\ J}$.
Joining drops makes less surface, so energy is released (it warms the water very slightly). Splitting a drop into many does the opposite: it needs energy (Exercise 22).

Check 4

A drop of water is sprayed into many tiny droplets. Compared with the drop, the droplets together have:

a the same surface area and the same surface energy
b more surface area, so work had to be done to make them
c less surface area, so energy was released
d more volume

## 3.4 Angle of contact

Why does water spread into a thin film on clean glass, while mercury on glass gathers into round, shiny beads? Why do raindrops roll off a lotus or rhododendron leaf instead of wetting it?

Two kinds of attraction compete:

- **Cohesion:** the attraction between molecules of the **same** substance (water to water). It holds a liquid together and pulls it into drops.
- **Adhesion:** the attraction between molecules of **different** substances (water to glass). It makes a liquid stick to a solid and spread over it.

Where a liquid's surface meets a solid, it is curved: this curve is the **meniscus**. The **angle of contact** $\theta$ is the angle between the tangent to the liquid surface at the point of contact and the solid surface, measured **inside the liquid** (Figure 3.14).

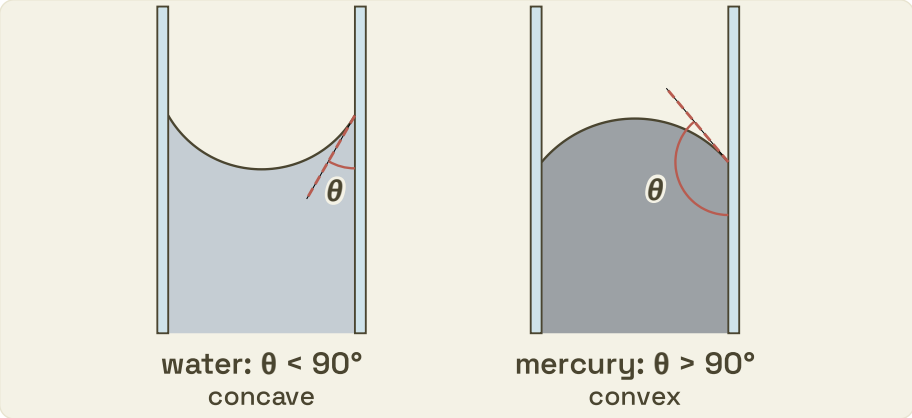



*Figure 3.14* *Liquids in glass tubes. Water (left) curves up the glass in a concave meniscus: its angle of contact $\theta$, measured inside the liquid between the glass and the tangent to the water surface, is less than 90°. Mercury (right) curves down from the glass in a convex meniscus, with $\theta$ greater than 90°.*

- If **adhesion is stronger than cohesion**, the liquid spreads over the solid and climbs it: the meniscus is **concave** (curving up at the edges) and $\theta$ is **less than 90°**. The liquid **wets** the solid. Water on clean glass has $\theta$ close to 0°; kerosene on glass, about 26°.
- If **cohesion is stronger than adhesion**, the liquid gathers itself up and pulls away from the solid: the meniscus is **convex** (curving down at the edges) and $\theta$ is **more than 90°**. The liquid **doesn't wet** the solid. Mercury on glass has $\theta \approx 140°$; water on wax or on a waxy leaf, more than 100°.
- $\theta = 90°$ exactly gives a flat surface right up to the wall (pure water on silver is close to this).

The angle of contact depends on:

1. the **nature of the solid and the liquid** (water–glass differs from mercury–glass);
2. the **cleanliness** of the surfaces (a trace of grease on glass makes water bead up);
3. the **medium above** the liquid's free surface (air or another gas or liquid);
4. the **temperature** of the liquid;
5. **impurities** dissolved in the liquid (a detergent lowers $\theta$, making water wet better).





1. Take a tube of flow of changing cross-section through which an incompressible, non-viscous fluid flows steadily (Figure 3.29). At end 1 the pressure is $P_1$, the area $A_1$, the height above a reference level $h_1$ and the speed $v_1$; at end 2 they are $P_2$, $A_2$, $h_2$ and $v_2$.
2. In a short time $\Delta t$, the fluid at end 1 moves a distance $s_1$ and at end 2 a distance $s_2$. The fluid is incompressible, so the volume entering equals the volume leaving:
$$A_1 s_1 = A_2 s_2 = \Delta V$$
3. The fluid behind end 1 pushes the fluid in the tube forward with a force $P_1 A_1$, through $s_1$: work done **on** the fluid, $W_1 = P_1 A_1 s_1 = P_1 \Delta V$.
4. At end 2 the fluid ahead pushes back with $P_2 A_2$, while the fluid moves forward $s_2$: work done **by** the fluid, so the work done on it is $W_2 = -P_2 A_2 s_2 = -P_2 \Delta V$.
5. The net work done on the fluid by the pressure:
$$W = (P_1 - P_2)\,\Delta V$$
6. The overall effect is that a slice of volume $\Delta V$ and mass $\rho \Delta V$ has been moved from end 1 to end 2 (the fluid in between is the same as before, in steady flow). Its kinetic energy has changed by $\frac{1}{2}\rho \Delta V (v_2^2 - v_1^2)$ and its potential energy by $\rho \Delta V g (h_2 - h_1)$.
7. With no viscosity, no energy is lost to friction, so by the work–energy theorem the work done equals the gain in energy:
$$(P_1 - P_2)\,\Delta V = \tfrac{1}{2}\rho \Delta V (v_2^2 - v_1^2) + \rho \Delta V g (h_2 - h_1)$$
8. Cancel $\Delta V$ and gather each end's terms on its own side:
$$P_1 + \tfrac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \tfrac{1}{2}\rho v_2^2 + \rho g h_2$$

Ends 1 and 2 were any two points, so

$$\boxed{P + \tfrac{1}{2}\rho v^2 + \rho g h = \text{constant}}$$

which is **Bernoulli's equation**.

For a **horizontal** tube, $h_1 = h_2$, and

$$P + \tfrac{1}{2}\rho v^2 = \text{constant}, \qquad P_1 - P_2 = \tfrac{1}{2}\rho (v_2^2 - v_1^2)$$

So if $P_1 > P_2$, then $v_2 > v_1$: **where the speed of a fluid is greater, its pressure is lower**, and the other way round. This is **Bernoulli's principle**. It isn't that low pressure is a cause and speed an effect, or the reverse: a fluid speeds up only if something pushes it from behind harder than from in front, so a fluid that has sped up has necessarily come from higher pressure into lower.

Check 11

Water flows steadily along a horizontal pipe that narrows in the middle. Compared with the wide part, in the narrow part the water's:

a speed and pressure are both greater
b speed is greater and pressure is lower
c speed is lower and pressure is greater
d speed and pressure are both the same

WORKED EXAMPLE

### Pressure in a narrowing pipe

Water flows along a horizontal pipe. At point 1 its speed is 1.5 m/s and the pressure $1.6 \times 10^5$ Pa. At point 2 the pipe's cross-section is a third of that at point 1. Find the speed and the pressure at point 2. ($\rho = 1000$ kg/m³)

Answer

Continuity: $v_2 = \dfrac{A_1}{A_2} v_1 = 3 \times 1.5 = \mathbf{4.5\ m/s}$.

Bernoulli, horizontal: $P_2 = P_1 - \frac{1}{2}\rho (v_2^2 - v_1^2) = 1.6 \times 10^5 - \frac{1}{2} \times 1000 \times (20.25 - 2.25) = 1.6 \times 10^5 - 9000 = \mathbf{1.51 \times 10^5\ Pa}$.

### Torricelli's theorem: water from a hole in a tank

A tank is filled with water to a height $h$ above a small hole in its side (or bottom). How fast does the water come out?

Apply Bernoulli's equation between point 1 on the water surface and point 2 just outside the hole:

1. Both are open to the air: $P_1 = P_2 = P_0$, the atmospheric pressure.

## 3.11 Bernoulli's equation and its applications

A STORY FROM SCIENCE

**Daniel Bernoulli**

**Daniel Bernoulli** (1700–1782) came from a Swiss family of mathematicians in Basel: his father Johann and his uncle Jacob are both famous for their mathematics, and the father, jealous, published his own book on the same subject and dated it years earlier, so that it seemed to come first. In his book, *Hydrodynamica* (1738), Daniel showed that where a moving fluid speeds up, its pressure drops. He also made the first kinetic theory of gases, explaining a gas's pressure by the collisions of its moving particles (Grade 11, 13.9 Pressure exerted by a gas). The equation in the form we use was written down by his friend Leonhard Euler.

**Bernoulli's theorem:** for an incompressible, non-viscous fluid in steady (streamline) flow, the total energy per unit volume (the sum of the pressure, the kinetic energy per unit volume and the potential energy per unit volume) is the same at every point along a streamline:




*Figure 3.28* *Daniel Bernoulli, whose Hydrodynamica (1738) gave the principle its name.*
Photo: Johann Jakob Haid / Johann Rudolf Huber, Public domain

$$P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$$

or, per unit mass, $\frac{P}{\rho} + \frac{1}{2}v^2 + gh = \text{constant}$.

### Derivation

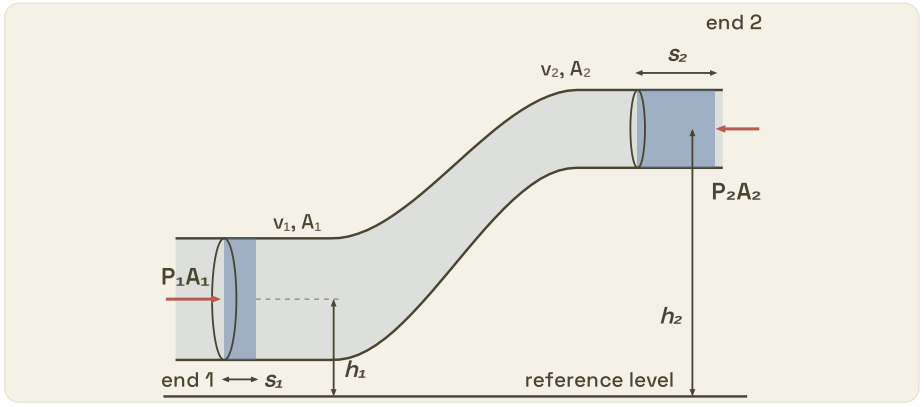



*Figure 3.29* *A tube of flow whose cross-section and height change. At end 1 the fluid has pressure $P_1$, speed $v_1$, area $A_1$, height $h_1$; at end 2, $P_2$, $v_2$, $A_2$, $h_2$. In a time $\Delta t$ the fluid between the two ends moves on: a slice of length $s_1$ enters at end 1 and an equal volume of length $s_2$ leaves at end 2.*





Check 5

A liquid whose cohesion is weaker than its adhesion to a solid has, on that solid:

a a concave meniscus and an angle of contact less than 90°
b a convex meniscus and an angle of contact more than 90°
c a flat surface and θ = 90°
d an angle of contact of exactly 180°

## 3.5 Capillarity

Dip a narrow glass tube (a **capillary tube**, from the Latin *capillus*, a hair) into water: the water climbs up inside it, higher than the water outside, and the narrower the tube, the higher it goes. Dip it into mercury and the mercury inside stands **lower** than outside.

**Capillarity** (capillary action) is the rise or fall of a liquid in a narrow tube or in the fine pores of a material, caused by surface tension and the adhesion between the liquid and the material. A liquid that wets the tube ($\theta < 90°$) rises; one that doesn't wet it ($\theta > 90°$) is pushed down.

Capillarity at work:

1. **The split in a fountain-pen nib** is a narrow slit. Ink wets the metal, and capillarity draws it along the slit to the tip, steadily, whichever way the pen is held.
2. **A towel** is a mass of fine fibers with narrow gaps between them, each like a capillary tube. Water wets the cotton and is drawn from your skin into the gaps.
3. **The wick of an oil lamp** (a diyo's cotton wick) is full of tiny gaps between its fibers. Oil wets the cotton and creeps up the wick to the flame.
4. **Soil and plants.** Water rises through the fine pores of soil by capillarity, which keeps the soil near the roots moist. In plants, capillarity helps water into the fine walls of cells and up the narrow tubes of the xylem, but only a little way: the main lift in a tall tree is the pull of water evaporating from the leaves (transpiration), which draws a continuous column of water up the xylem.
5. **Blotting paper, a sponge, a sugar cube** dipped in tea: all soak liquid up through their pores.
6. **Damp walls.** Water climbs from wet ground up into a brick wall through its pores; a waterproof layer (a damp-proof course) near the bottom of the wall stops it.

CAREFUL

**What capillarity doesn't do**

The tiny blood vessels called **capillaries** are named for being as fine as hairs, but blood is driven through them by the **heart's pumping**, not by capillary action. And the liquid in a **thermometer** rises because it **expands** when heated: the narrow bore only makes a small expansion show as a long movement. Neither is an example of capillarity.

### Capillary rise: the formula

1. A capillary tube of radius $r$ stands in a liquid of density ρ and surface tension $T$, which wets it with an angle of contact θ (Figure 3.15). The liquid rises to a height $h$ above the level outside.
2. The liquid surface meets the glass all round a circle of circumference $2\pi r$. There, the liquid's surface pulls on the glass with a force $T$ per unit length, along the surface's tangent, down and away from the wall. By Newton's third law, the glass pulls the liquid with an equal and opposite force: up, at an angle θ to the vertical wall.
3. The horizontal parts of these forces cancel all the way round. The vertical part is
$$F = 2\pi r \times T\cos\theta$$
4. This holds up the weight of the column of liquid raised above the outside level. Its volume is $\pi r^2 h$ (ignoring the small amount in the meniscus itself), so its weight is
$$W = \pi r^2 h \rho g$$
5. In equilibrium, $F = W$:
$$2\pi r T\cos\theta = \pi r^2 h\rho g$$
$$h = \frac{2T\cos\theta}{r\rho g} \quad \text{or} \quad T = \frac{r h \rho g}{2\cos\theta}$$

What it says:

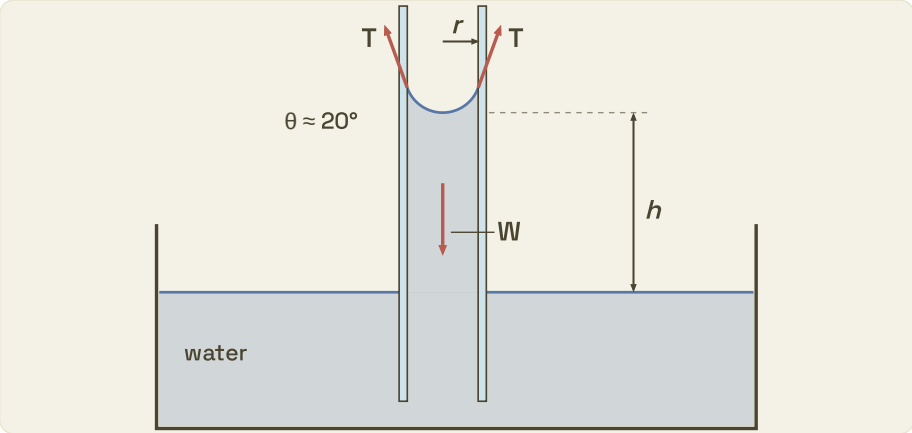



*Figure 3.15* *A capillary tube of radius r dipped in water. The water rises to a height h above the outside level. All round the rim of the meniscus, the glass pulls the liquid with a force T per unit length, at the angle of contact θ to the wall (the reaction to the liquid surface's pull on the glass): the upward part of this pull, $2\pi r T\cos\theta$, holds up the weight of the raised column.*

- $h \propto \dfrac{1}{r}$: a tube half as wide makes the liquid rise twice as high (**Jurin's law**). So $hr$ is the same for any tube with a given liquid: $h_1 r_1 = h_2 r_2$.
- $h \propto \dfrac{1}{g}$: on the Moon, where $g$ is a sixth of Earth's, the same tube would hold a column six times as tall.
- For $\theta < 90°$, $\cos\theta$ is positive and $h$ is positive: the liquid **rises**. For $\theta > 90°$ (mercury in glass), $\cos\theta$ is negative: $h$ is negative, a **fall** (capillary depression). For $\theta = 90°$, no rise or fall.
- For water in clean glass, $\theta \approx 0°$ and $\cos\theta \approx 1$: $h = \dfrac{2T}{r\rho g}$.

DID YOU KNOW?

**A tube that is too short doesn't overflow**

If the water would rise 6 cm but the tube sticks only 4 cm out of the water, does water spill out of the top and make a fountain that runs forever? No. The water climbs to the top and stops there: the meniscus flattens (its radius of curvature grows) until the upward pull of the surface balances the weight of the 4 cm column. Capillarity can never make a perpetual-motion machine.

Check 6

In a capillary tube of radius 1 mm, water rises 1.5 cm. In a tube of radius 0.5 mm of the same glass, it rises:

a 0.75 cm
b 1.5 cm
c 3.0 cm
d 6.0 cm



WORKED EXAMPLE

**Blood through a narrowed artery**

Blood flows at 0.4 m/s where an artery has a radius of 3 mm. Farther along, a blockage narrows it to a radius of 1.5 mm.

a. Find the speed of the blood in the narrowed part.
b. By what factor has the speed increased?

Answer

$v_1 = 0.4$ m/s, $r_1 = 3\text{ mm} = 3\times10^{-3}$ m, $r_2 = 1.5\text{ mm} = 1.5\times10^{-3}$ m.
$A_1 = \pi r_1^2 = \pi\times(3\times10^{-3})^2 = 9\pi\times10^{-6}$ m²; $A_2 = \pi r_2^2 = 2.25\pi\times10^{-6}$ m².
a. $A_1 v_1 = A_2 v_2$, so

$$v_2 = \frac{A_1 v_1}{A_2} = \frac{9\pi\times10^{-6}\times0.4}{2.25\pi\times10^{-6}} = \mathbf{1.6\ m/s}$$

b. $\dfrac{v_2}{v_1} = \dfrac{1.6}{0.4} = \mathbf{4}$. Halving the radius quarters the area, so the speed is four times as great: $v \propto 1/r^2$ for a round pipe.

YOUR TURN

**Filling a tank**

A pipe that narrows from one end to the other fills a 500-liter tank. At the wide end its cross-section is 50 cm², and water enters at 1.2 m/s; at the narrow end, where the water pours into the tank, it is 10 cm².

a. Find the speed of the water at the narrow end.
b. How long does the tank take to fill, if the flow is steady?

Answer

a. $v_2 = \dfrac{A_1 v_1}{A_2} = \dfrac{50\times1.2}{10} = \mathbf{6\ m/s}$ (the areas' units cancel, so cm² can stay).
b. Rate of flow $= A_1 v_1 = 50\times10^{-4}\times1.2 = 6\times10^{-3}$ m³/s, which is 6 liters a second. The tank holds 500 L = 0.5 m³, so the time is $\dfrac{0.5}{6\times10^{-3}} = 83$ s, **1 minute 23 seconds**.

Check 10

Water flows at 3 m/s through a pipe 4 cm across, which then narrows to 2 cm across. Its speed in the narrow part is:

a 1.5 m/s
b 6 m/s
c 12 m/s
d 24 m/s

## Energy of a flowing fluid

A fluid in steady flow can have three kinds of energy:

1. **Kinetic energy**, because it moves: for a mass $m$ moving at $v$, $KE = \frac{1}{2}mv^2$. Per unit mass: $\frac{1}{2}v^2$. A unit volume has a mass ρ, so per unit volume: $\frac{1}{2}\rho v^2$.
2. **Potential energy**, because of its height $h$ above a reference level: $PE = mgh$. Per unit mass: $gh$; per unit volume: $\rho gh$.
3. **Pressure energy**, because it is under pressure: the work that was done to push it into place against the pressure of the fluid already there.

**Pressure energy.** Take a tube of uniform cross-section $A$ holding an incompressible, non-viscous fluid at a pressure $P$. Push a small extra mass Δm of fluid into it slowly (so that it gains no kinetic energy). It fills a length Δl of the tube, so $\Delta m = A\,\Delta l\,\rho$. The force needed is $PA$, through a distance Δl, so the work done is $PA\,\Delta l$. This work is stored as the pressure energy of the mass Δm. Per unit mass:

$$\frac{PA\,\Delta l}{\Delta m} = \frac{PA\,\Delta l}{A\,\Delta l\,\rho} = \frac{P}{\rho}$$

and per unit volume, $\dfrac{P}{\rho}\times\rho = P$. So **pressure is the pressure energy per unit volume**: 1 Pa = 1 N/m² = 1 J/m³.

Altogether, the total energy of a flowing fluid is

$$\text{per unit mass: } \frac{1}{2}v^2 + gh + \frac{P}{\rho} \qquad \text{per unit volume: } \frac{1}{2}\rho v^2 + \rho gh + P$$

where $A_1$, $A_2$ are the cross-sectional areas at two places along the pipe (or a tube of flow) and $v_1$, $v_2$ the fluid's speeds there. **Where the pipe narrows, the fluid speeds up; where it widens, the fluid slows down.**

### Derivation

The fluid is taken as **incompressible** (its density ρ is the same everywhere), the flow as **steady** (at any point the velocity doesn't change with time), and, for simplicity, the fluid as **non-viscous**, so that it moves with one speed across each section.

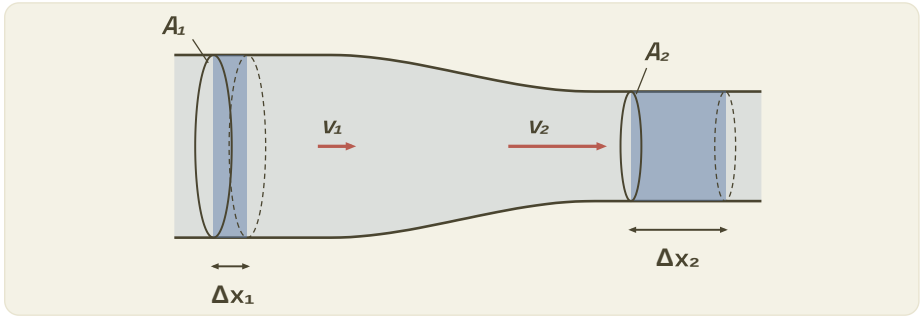



*Figure 3.27* *A tube of flow whose cross-section changes from $A_1$ to $A_2$. In a time Δt, the fluid at the first end moves on by $\Delta x_1 = v_1\Delta t$, and at the other end by $\Delta x_2 = v_2\Delta t$: the two shaded slices have the same volume.*

1. Take a narrow tube of flow (a bundle of streamlines, which fluid never crosses) with cross-sections $A_1$ and $A_2$ at its two ends, where the fluid moves with speeds $v_1$ and $v_2$ (Figure 3.27).
2. In a short time Δt, the fluid entering at the first end moves a distance $\Delta x_1 = v_1\Delta t$, so the volume entering is $A_1 v_1 \Delta t$ and its mass $\rho A_1 v_1 \Delta t$.
3. In the same time the volume leaving at the other end is $A_2 v_2 \Delta t$, of mass $\rho A_2 v_2 \Delta t$.
4. Fluid can't pile up in the tube or appear from nowhere in steady flow (and it can't be squeezed, since it is incompressible), so the mass entering equals the mass leaving:

$$\rho A_1 v_1 \Delta t = \rho A_2 v_2 \Delta t \quad \Rightarrow \quad A_1 v_1 = A_2 v_2$$

So **Av is the same all along the tube**:

$$Av = \text{constant} \qquad \text{so} \qquad v \propto \frac{1}{A}$$

The product $Av$ is the volume of fluid passing any cross-section per second, the **volume rate of flow** (rate of flow, rate of discharge, or volume flux), in m³/s. The speed of the fluid at any section is **inversely proportional** to the area of the section there.

(For a compressible fluid, such as a gas at high speed, it is the mass per second, $\rho Av$, that stays the same.)

### Applications

1. **Water pipes and hoses.** It tells how the speed changes where a pipe narrows (faster) or widens (slower), and lets an engineer size a pipe for a given flow. A hose's nozzle makes a fast jet.
2. **Blood flow.** Blood speeds up where an artery is narrowed by a deposit. And the other way: the aorta is about 2.5 cm across and its blood moves at about 30 cm/s, but the body's billions of capillaries together have a cross-section hundreds of times larger, so blood creeps through them at under a millimeter a second, slowly enough to exchange oxygen and food with the cells.
3. **Ventilation.** Air ducts are sized so that air moves at a steady, quiet speed when the duct changes size.
4. **Jet engines and rocket nozzles.** A narrowing nozzle speeds up the hot gas rushing out of a jet engine. (Gases at very high speed are compressible, so engineers use the mass flow ρAv; a rocket nozzle, where the gas is faster than sound, widens again after its throat.)
5. **Pumps, turbines and hydropower.** The water entering a turbine is sped up through a narrowing nozzle or guide vanes before it strikes the blades.
6. **Rivers and wind.** A river runs fastest through a narrow gorge and slows where its valley widens (the Kali Gandaki at its narrowest is fierce). Wind speeds up through a mountain pass or between tall buildings.





WORKED EXAMPLE

**Oil and water in a U-tube**

A U-tube open at both ends contains water. Oil, which doesn't mix with water, is poured into one arm until the water in the other arm has risen 65 mm above its first level, and the top of the oil stands 10 mm above the water surface in that other arm (Figure 3.16). Find the density of the oil. (ρ of water = 1000 kg/m³)

a. 929 kg/m³
b. 650 kg/m³
c. 425 kg/m³
d. 800 kg/m³

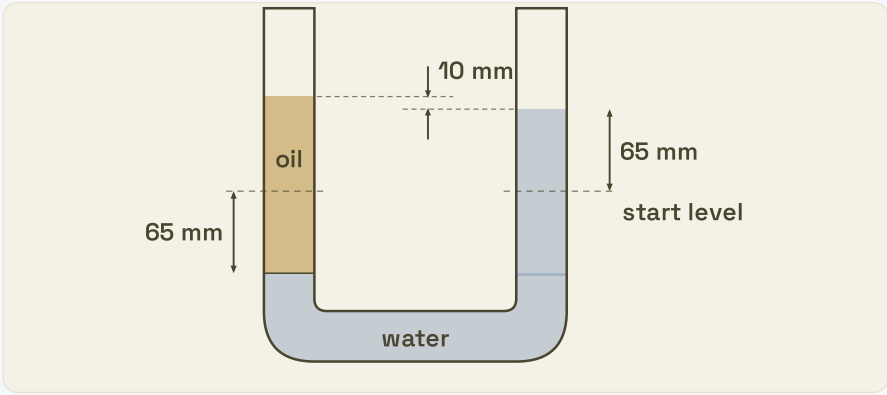



*Figure 3.16* *Oil in the left arm pushes the water down 65 mm there and up 65 mm in the right arm. The oil's top stands 10 mm above the water surface in the right arm.*

Answer

The water in the oil arm has gone **down** 65 mm and in the other arm **up** 65 mm, so the oil–water boundary is 65 + 65 = 130 mm below the water surface in the right arm.

Take the level of the boundary. At that level the pressure is the same in both arms (the water below it is one connected body at rest). Above it, the left arm holds the oil column, of height $h_0 = 130 + 10 = 140$ mm; the right arm holds a water column of height $h_w = 130$ mm. Both arms are open to the same air pressure, so

$$h_0\rho_0 g = h_w\rho_w g$$

$$\rho_0 = \frac{h_w}{h_0}\rho_w = \frac{130}{140} \times 1000 = 928.6 \approx \mathbf{929\ kg/m^3}$$

The answer is **a**. The oil column is taller than the water column it balances because oil is less dense.

## 3.6 Viscosity

Now the fluids start to move. **Fluid dynamics** studies fluids in motion, and its first idea is that a moving fluid has internal friction.

A STORY FROM SCIENCE

**Newton and the "lack of slipperiness"**

**Isaac Newton** (1643–1727), whose laws of motion you used for rotation in Unit 1, also studied fluids. In his *Principia* (1687) he supposed that the resistance between two layers of a fluid sliding over each other grows in proportion to how fast one slides past the other: he called it a "lack of slipperiness". Fluids that obey this rule exactly (water, air, most oils) are still called **Newtonian fluids**. Others don't: ketchup flows only once you shake it, and a cornflour-and-water paste goes hard when you hit it.

Three things you have seen:

1. Stir water in a bowl and stop: the swirl slows down and dies away.
2. In a river, the water at the surface flows fastest, and slower and slower towards the bed.
3. In a pipe, water moves fastest along the middle and slowest near the walls.

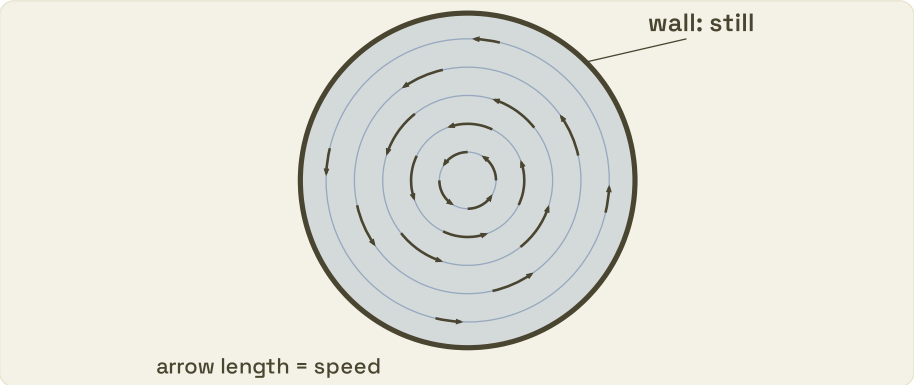



*Figure 3.17* *Water stirred in a beaker, seen from above. Each ring of water drags on the rings beside it; the slower layer next to the wall drags on the faster layer inside it, and the swirl dies away.*

When you stop stirring, every layer of water rubs against the layers next to it. The faster layer is dragged back by the slower one beside it (and the slower one is dragged forward). Near the walls, the water touching the bowl doesn't move at all, and it holds back the layer next to it. This internal friction between layers turns the swirl's kinetic energy into heat, until the water is still.

In a river the same thing happens from the bottom up (Figure 3.18). The water touching the bed is held still by it (a fluid in contact with a solid moves with the solid: the **no-slip condition**). The layer just above slides over that still layer and is held back by it; the layer above that slides over a slow layer and is held back a little less; and so on. Each layer is dragged back by the slower layer below it and dragged forward by the faster one above, so the speed grows steadily from zero at the bed to its greatest at the surface, where nothing above holds it back.

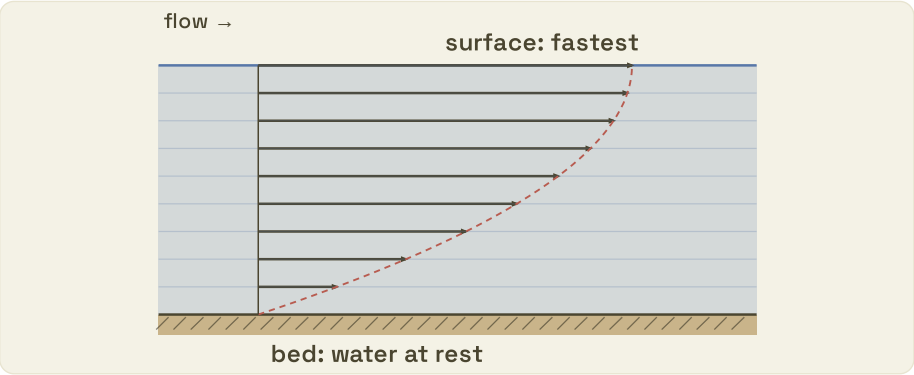



*Figure 3.18* *Water flowing in thin layers over a river bed: still at the bed, faster in each layer above, fastest at the surface. The arrows' lengths are the layers' speeds.*

In a pipe, the water touching the wall is still, and the speed rises to its greatest along the axis: the water flows in thin cylindrical sheets, one inside another, called **laminae** (Figure 3.19). The change of speed from layer to layer is the **velocity gradient**.

**Viscosity** is the property of a fluid by which it opposes the relative motion between its layers. The opposing force between layers is the **viscous drag** (viscous force). It is like friction between solid surfaces, but it acts inside the fluid, between one layer and the next. The more viscous a liquid, the more it resists flowing: honey is far more viscous than water.

**Temperature:**

- The viscosity of a **liquid decreases** as it warms, quickly: honey runs when warm, and water's viscosity halves between 20 °C and 60 °C. The molecules move faster and pull away from one another more easily. This is why machines are hard to start on





6. **Soil and geology.** Shake soil in water and let it settle: the sand settles first, then silt, then clay. Timing it gives the soil's particle sizes (its texture).
7. **Millikan's oil drop experiment** used Stokes' law to find the size of tiny charged oil drops from how fast they fell, and so measured the charge of the electron.

DID YOU KNOW?

**Why clouds don't fall**

A cloud droplet is about 0.01 mm across. Its terminal velocity in air, from $v = \dfrac{2r^2(\rho - \sigma)g}{9\eta}$ with $r = 5 \times 10^{-6}$ m, $\rho =$ 1000 kg/m³ and $\eta = 1.8 \times 10^{-5}$ Pa s, is about 3 mm/s, about 10 m an hour. The rising air that makes clouds moves faster than that, so the droplets are carried up or held. Only when droplets join into raindrops about a million times heavier do they fall out as rain. A parachute and a hailstone also reach terminal velocities, though at their size and speed the drag grows more like the square of the speed than in Stokes' law.

Check 9

Two steel balls, of radii 1 mm and 2 mm, fall through the same glycerine. The ratio of their terminal velocities (small : large) is:

a 1 : 2 b 1 : 4
c 1 : 8 d 2 : 1

WORKED EXAMPLE

**A settling tank**

In a tank for separating plastic beads from a liquid, the beads (density 1050 kg/m³) must settle at a terminal velocity of at least 2.0 mm/s through a liquid of viscosity 1.2 N s/m² and density 1000 kg/m³. What is the smallest radius the beads can have? (g = 9.8 m/s²)

Answer

From $v = \dfrac{2r^2(\rho - \sigma)g}{9\eta}$, $r = \sqrt{\dfrac{9\eta v}{2(\rho - \sigma)g}}$.

With $\eta = 1.2$ N s/m², $v = 2.0 \times 10^{-3}$ m/s, $\rho = 1050$ kg/m³, $\sigma = 1000$ kg/m³:

$$r = \sqrt{\frac{9 \times 1.2 \times 2.0 \times 10^{-3}}{2 \times (1050 - 1000) \times 9.8}} = \sqrt{\frac{0.0216}{980}} = \sqrt{2.20 \times 10^{-5}} = 4.69 \times 10^{-3} \text{ m}$$

The beads must be at least **4.69 mm** in radius. Smaller beads settle more slowly.

YOUR TURN

**A bead in glycerine**

A tiny metal bead of density 8900 kg/m³ falls through glycerine of density 1260 kg/m³ and viscosity 1.5 N s/m². What must its diameter be for a terminal velocity of exactly 3.5 mm/s? (g = 9.8 m/s²)

Answer

$$r = \sqrt{\frac{9\eta v}{2(\rho - \sigma)g}} = \sqrt{\frac{9 \times 1.5 \times 3.5 \times 10^{-3}}{2 \times 7640 \times 9.8}} = \sqrt{\frac{0.04725}{1.497 \times 10^{5}}} = \sqrt{3.16 \times 10^{-7}} = 5.62 \times 10^{-4} \text{ m}$$

Diameter = 2r = **1.12 mm**.

## 3.10 Equation of continuity and its applications

Put your thumb partly over the end of a garden hose and the water shoots out much faster and farther. The same water has to get through a smaller opening each second, so it must move faster.

The **equation of continuity** is the law of conservation of mass for a flowing fluid. For an **incompressible** fluid in **steady** flow, the volume of fluid entering a pipe each second equals the volume leaving it each second, so

$$A_1 v_1 = A_2 v_2$$

At first the ball is slow and the drag small, so it speeds up. As its speed grows, so does the drag, and its acceleration gets less and less. Soon the drag and upthrust together balance its weight: the net force is zero and the ball falls on at a **steady speed**, its **terminal velocity** (or settling velocity).

**Derivation.** Let the sphere have radius $r$ and density ρ, and the liquid density σ and viscosity η.

1. Weight: $W = \text{volume} \times \rho \times g = \frac{4}{3}\pi r^3 \rho g$
2. Upthrust (the weight of liquid displaced): $U = \frac{4}{3}\pi r^3 \sigma g$
3. Viscous drag at the terminal velocity $v$: $F = 6\pi\eta r v$
4. At the terminal velocity the net force is zero: $F + U = W$, so
$$6\pi\eta r v = \tfrac{4}{3}\pi r^3 \rho g - \tfrac{4}{3}\pi r^3 \sigma g = \tfrac{4}{3}\pi r^3 (\rho - \sigma) g$$
5. Dividing by $6\pi\eta r$:
$$v = \frac{2r^2(\rho - \sigma)g}{9\eta}$$

What it says:

- $v \propto r^2$: a sphere twice the radius falls four times as fast. A graph of $v$ against $r^2$ is a straight line through the origin (Figure 3.26).
- If ρ > σ the sphere falls; if ρ < σ, $v$ is negative: it **rises** (an air bubble in a fizzy drink, a drop of oil in water).
- A more viscous liquid gives a slower fall.

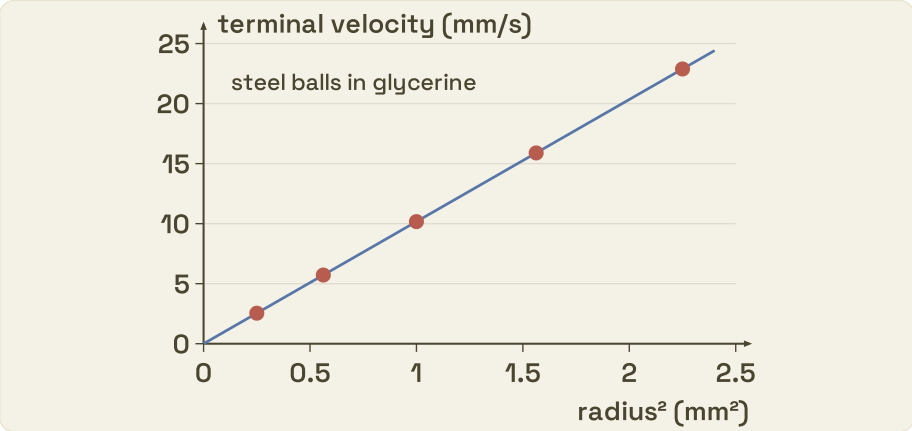



*Figure 3.26* *Terminal velocity against the square of the radius for steel balls falling through glycerine: a straight line through the origin, as $v = \frac{2(\rho-\sigma)g}{9\eta} r^2$ says.*

## Applications

1. **Measuring the viscosity of a liquid (the falling-sphere method).** Time a small ball of known radius and density over a marked distance in a tall jar of the liquid, once it has reached its terminal velocity. Rearranging,
$$\eta = \frac{2r^2(\rho - \sigma)g}{9v}$$
2. **Settling and sedimentation.** How fast mud settles in a water-treatment tank, or silt in a reservoir behind a dam, depends on the particles' size: fine clay takes days, sand seconds.
3. **ESR in medicine.** The erythrocyte sedimentation rate is how many millimeters red blood cells settle in an hour in a tube of blood. Some illnesses make the cells clump into larger groups, which settle faster, so a high ESR tells the doctor to look further.
4. **Paints, emulsions and cosmetics** are made with small particles and a viscous liquid so that they settle slowly and stay mixed.
5. **Cleaning the air.** Settling chambers and cyclones, which remove dust from factory smoke, are designed from settling speeds.





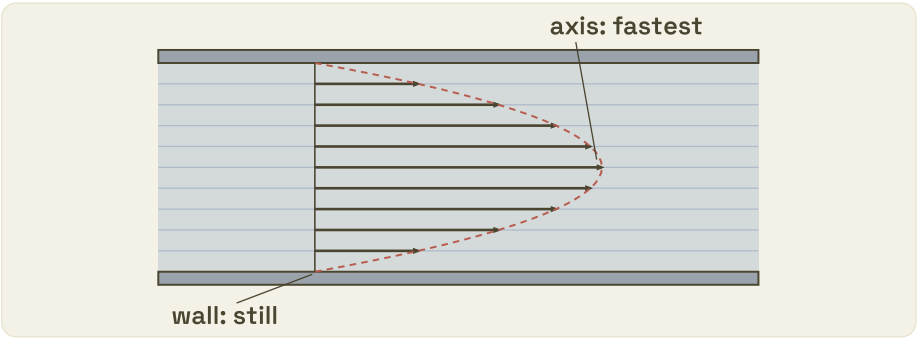



*Figure 3.19* *Smooth flow in a pipe: thin cylindrical layers slide inside one another, the one at the wall still and the one along the axis fastest. The tips of the speed arrows lie on a parabola.*

a cold winter morning: the oil and grease in them thicken, and the moving parts drag.

- The viscosity of a **gas increases** as it warms. A gas's "friction" comes from its molecules flying from one layer to the next carrying their momentum with them, and faster molecules carry more.

**Check 7**

Engine oil for a motorbike in Jumla's winter should be:

a more viscous than oil for summer, since cold makes oil thin

b less viscous at low temperatures than summer oil, since cold makes oil thick

c the same as summer oil, since viscosity doesn't depend on temperature

d replaced by water, which never thickens

## Streamline, laminar and turbulent flow

**Streamline (steady) flow.** Watch a fluid at one point, A. If every particle that passes A has the same velocity there (the same speed in the same direction) as the ones before it, and the same is true at every point, the flow is **steady**, or **streamline flow**. Each particle then follows the same path as the one before it through A, B and C, with the velocity $v_1$ at A, $v_2$ at B and $v_3$ at C each time. That path is a **streamline**: a line whose tangent at any point is the direction of the fluid's velocity there (Figure 3.20).

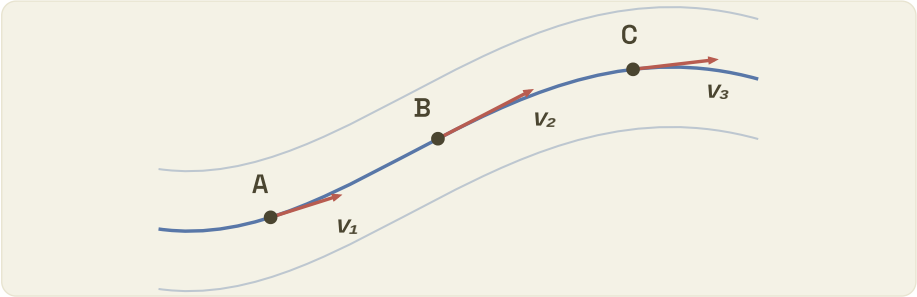



*Figure 3.20* *A streamline through the points A, B and C. At each point the velocity of the fluid ($v_1$, $v_2$, $v_3$) is along the tangent to the streamline, and in steady flow it is the same for every particle that passes that point.*

Two streamlines can never cross: if they did, a particle at the crossing point would have two directions to go in at once.

**Laminar flow.** If a liquid flows steadily in **layers** sliding over one another with different speeds, without mixing, the flow is **laminar flow**. It is streamline flow seen as layers (the river and the pipe above).

**Turbulent flow.** Turn a tap on slowly and the stream is clear and smooth, like a glass rod. Open it wide and it breaks into a rough, churning, white rope. Above a certain speed, the **critical velocity**, the flow becomes **turbulent**: the particles move in eddies and whirls, and the speed and direction at any point keep changing irregularly from moment to moment. The water of the Bhote Koshi tumbling over its boulders, smoke rising from an incense stick that is smooth for a few centimeters and then breaks into swirls, the air behind a fast bus: all are turbulent.

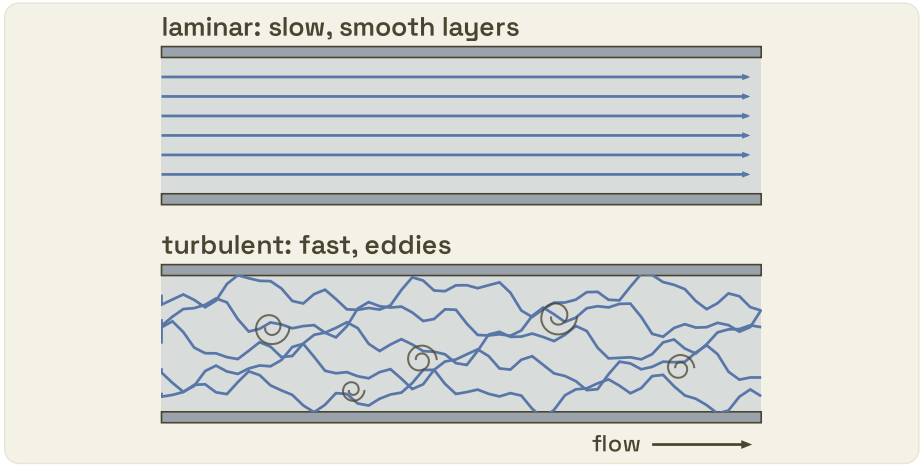



*Figure 3.21* *Above: laminar flow in a pipe, every layer sliding straight along. Below: turbulent flow at a higher speed: the fluid churns in eddies while it moves along on average.*

| Laminar (streamline) flow | Turbulent flow |
|---|---|
| Below the critical velocity | Above the critical velocity |
| Smooth layers that don't mix | Eddies that mix the fluid |
| The velocity at a point stays the same | The velocity at a point keeps changing |
| Viscous drag on a moving body grows in proportion to its speed | Drag grows roughly as the square of the speed |
| Blood in small vessels, honey poured slowly, oil in a bearing | A fast river, a jet from a hose, the air behind a truck |

### Reynolds number

What decides whether a flow is laminar or turbulent? In 1883 the British engineer **Osborne Reynolds** let a thin stream of dye into water flowing through a glass pipe. At low speed the dye stayed a sharp thread; at higher speed it suddenly broke up and filled the pipe. He found that the change depends not on the speed alone, but on one number made from four quantities, now called the **Reynolds number**:

$$R_e = \frac{\rho v D}{\eta}$$

where ρ is the fluid's density, $v$ its average speed, $D$ the diameter of the pipe and η the fluid's viscosity (Section 3.7). It has no unit: it compares the fluid's tendency to keep moving (its inertia, ρv) with the viscous drag that tames it (η/D). For flow in a straight pipe:

- $R_e$ below about **2000**: laminar flow;
- $R_e$ above about **3000**: turbulent flow;
- in between: unsteady, changing from one to the other.

Setting $R_e = 2000$ gives the **critical velocity**, $v_c = \dfrac{2000\,\eta}{\rho D}$: a viscous fluid in a narrow pipe stays laminar up to a high speed, while water in a wide pipe turns turbulent at a crawl.





$$F = 6\pi\eta r v$$

This is **Stokes' law**. It holds for a **small, smooth, rigid sphere moving slowly** through a fluid that stretches far around it, so that the flow around it stays laminar (for a Reynolds number $\rho v(2r)/\eta$ well below 1). A falling raindrop, a parachute or a cricket ball is too big and too fast: their drag grows more like $v^2$.

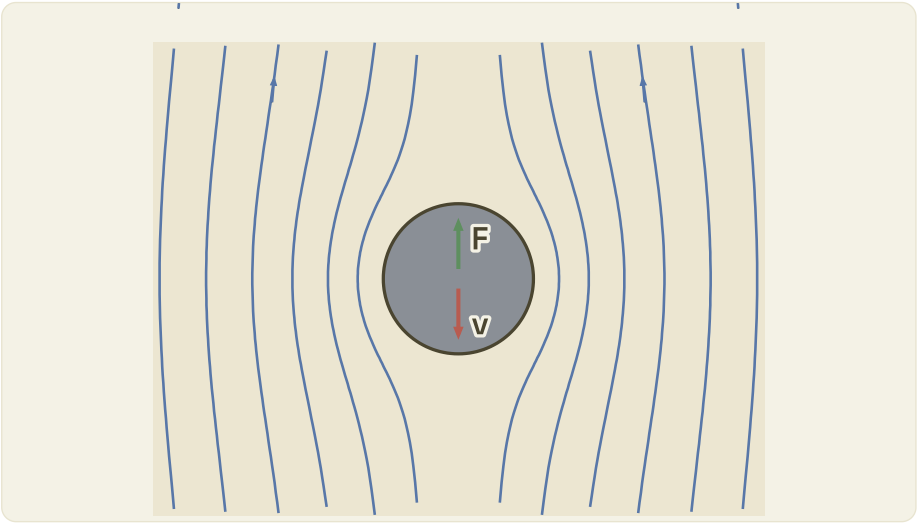



*Figure 3.24* *A small sphere falling slowly through a viscous liquid, seen as if moving down with it: the liquid streams up past it, smoothly, in streamlines (worked out from Stokes' theory). The liquid drags the sphere back with a force $F = 6\pi\eta r v$.*

The form of Stokes' law can be found by dimensions too: if $F = k\eta^a r^b v^c$, then $[\mathrm{MLT}^{-2}] = [\mathrm{ML}^{-1}\mathrm{T}^{-1}]^a[\mathrm{L}]^b[\mathrm{LT}^{-1}]^c$. M gives $a = 1$; T gives $-2 = -a - c$, so $c = 1$; L gives $1 = -a + b + c$, so $b = 1$. Then $F = k\eta r v$, and Stokes' theory gives $k = 6\pi$.

### Terminal velocity

Drop a small steel ball into a tall jar of glycerine. Three forces act on it (Figure 3.25):

1. its **weight** $W$, downwards;
2. the **upthrust** $U$ of the liquid, upwards (Archimedes);
3. the **viscous drag** $F$, upwards, against its motion, growing as the ball speeds up.

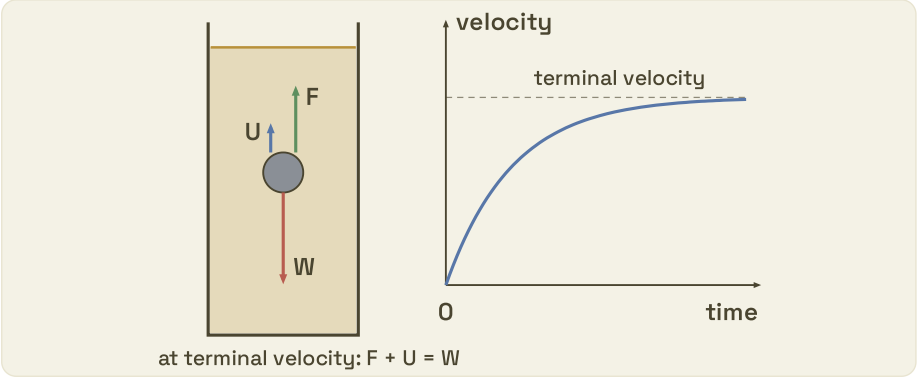



*Figure 3.25* *Left: the forces on a sphere falling through a liquid: weight W down, upthrust U and viscous drag F up. Right: its velocity against time: it speeds up, but more and more slowly, and levels off at the terminal velocity, when F + U = W.*

Check 8

A blockage halves the radius of a blood vessel. If the pressure difference stays the same, the flow through it becomes:

a half
b a quarter
c an eighth
d a sixteenth

WORKED EXAMPLE

**Widening a pipe**

A fluid of viscosity 0.001 Pa s flows through a narrow pipe 0.5 m long and 1 mm in radius, with a pressure difference of 2000 Pa across it.

a. Calculate the volume flow rate.
b. The radius is doubled, everything else staying the same. By what factor does the flow rate change? Find the new flow rate.

Answer

a. $P = 2000$ Pa, $r = 1$ mm $= 1 \times 10^{-3}$ m, $\eta = 0.001$ Pa s, $l = 0.5$ m.

$$V = \frac{\pi P r^4}{8\eta l} = \frac{\pi \times 2000 \times (10^{-3})^4}{8 \times 0.001 \times 0.5} = \frac{6.28 \times 10^{-9}}{0.004} = \mathbf{1.57 \times 10^{-6}\ m^3/s}$$

b. $r' = 2 \times 10^{-3}$ m. Since $V \propto r^4$, the flow changes by a factor $2^4 = \mathbf{16}$:

$$V' = 16 \times 1.57 \times 10^{-6} = \mathbf{2.51 \times 10^{-5}\ m^3/s}$$

DID YOU KNOW?

**Would Poiseuille really hold for the wider pipe?**

Check the Reynolds numbers. In the narrow pipe the average speed is $V/\pi r^2 = 1.57 \times 10^{-6}/(\pi \times 10^{-6}) = 0.5$ m/s, and $R_e = \rho v D/\eta = 1000 \times 0.5 \times 0.002/0.001 = 1000$: laminar, so the formula holds. In the wider pipe the average speed is 2 m/s and $R_e = 1000 \times 2 \times 0.004/0.001 = 8000$: turbulent. A real pipe would carry noticeably less than the formula's answer. Always ask whether a formula's assumptions hold before trusting its number.

YOUR TURN

**Two tubes**

Two tubes, A and B, carry the same liquid ($\eta = 0.002$ Pa s) under the same pressure difference. Tube A is 0.2 m long with a radius of 2 mm; tube B is 0.1 m long with a radius of 1.5 mm. Which delivers more liquid per second, and how many times more?

Answer

With the same $P$ and $\eta$, $V \propto \dfrac{r^4}{l}$:

$$\frac{V_A}{V_B} = \frac{r_A^4/l_A}{r_B^4/l_B} = \frac{(2)^4/0.2}{(1.5)^4/0.1} = \frac{80}{50.6} = \mathbf{1.58}$$

Tube A carries about 1.58 times as much, though it is twice as long: its larger radius, raised to the fourth power, wins.

## 3.9 Stokes' law

Some things you may have wondered about:

1. Clouds float in the sky, though they are made of water droplets, which are heavier than air.
2. A parachutist lands safely after jumping from a plane.
3. Hailstones falling from a great height don't hit as hard as a stone dropped from that height would without air.

Each is a body falling through a fluid whose resistance grows with its speed, until it stops speeding up.

### The law

In 1851 the Irish-born physicist **George Gabriel Stokes** worked out the viscous drag on a small sphere moving slowly through a fluid. The drag $F$ is directly proportional to the radius $r$ of the sphere, the coefficient of viscosity $\eta$ of the fluid, and the velocity $v$ of the sphere:





WORKED EXAMPLE

**Is the flow in a water pipe laminar?**

Water ($\rho = 1000$ kg/m$^3$, $\eta = 1.0 \times 10^{-3}$ Pa s) flows at 0.50 m/s through a household pipe 2.0 cm across. Is the flow laminar? What is the critical velocity for this pipe?

Answer

$$R_e = \frac{\rho v D}{\eta} = \frac{1000 \times 0.50 \times 0.020}{1.0 \times 10^{-3}} = 10{,}000$$

That is far above 3000: the flow is **turbulent**.

Critical velocity: $v_c = \dfrac{2000\eta}{\rho D} = \dfrac{2000 \times 1.0 \times 10^{-3}}{1000 \times 0.020} = \mathbf{0.10\ m/s}$, a slow trickle. Nearly all flow in household pipes is turbulent; laminar flow belongs to narrow tubes, slow flows and viscous liquids.

## 3.7 Newton's formula for viscosity; coefficient of viscosity

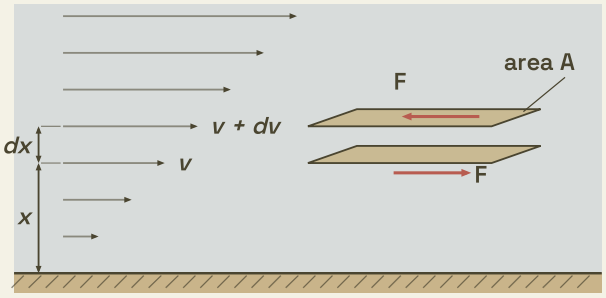



*Figure 3.22* Layers of liquid flowing over a fixed bed. Two layers, each of area $A$, are at heights $x$ and $x + dx$ above the bed, moving with velocities $v$ and $v + dv$. The faster layer is dragged back by a force $F$, the slower one pulled forward.

Take two neighboring layers of a liquid in laminar flow, each of area $A$, at heights $x$ and $x + dx$ above a fixed surface (the river bed, where the liquid is at rest), moving with velocities $v$ and $v + dv$ (Figure 3.22). The change of velocity per unit distance across the flow, $\dfrac{dv}{dx}$, is the **velocity gradient** between them.

Newton found that the viscous force $F$ between the layers is

- directly proportional to the **area** of the layers, $F \propto A$, and
- directly proportional to the **velocity gradient**, $F \propto \dfrac{dv}{dx}$.

Together, $F \propto A\dfrac{dv}{dx}$, or

$$F = -\eta A \frac{dv}{dx}$$

This is **Newton's formula (law) for viscosity**. The constant of proportionality, $\eta$ (eta), is the **coefficient of viscosity** of the fluid: a property of the fluid itself (and its temperature). The minus sign says that the viscous force on a layer acts **against** its motion relative to the layer next to it. When only its size is needed, $F = \eta A \dfrac{dv}{dx}$.

If $A = 1$ m$^2$ and $\dfrac{dv}{dx} = 1$ s$^{-1}$, then $F = \eta$ (in size). So the **coefficient of viscosity** of a fluid is the tangential viscous force needed to keep a unit velocity gradient between two layers, per unit area of the layers.

**Unit:** $\eta = \dfrac{F}{A\,(dv/dx)}$, so its SI unit is $\dfrac{\text{N}}{\text{m}^2 \times \text{s}^{-1}} = \text{N s/m}^2 = \text{Pa s}$ (pascal second), also called the **poiseuille** (Pl) or **decapoise**. In the CGS system the unit is the dyne s/cm$^2$, the **poise** (P), after Poiseuille:

$$1 \text{ poise} = 1\frac{\text{dyne s}}{\text{cm}^2} = \frac{10^{-5}\text{ N s}}{10^{-4}\text{ m}^2} = 0.1 \text{ N s/m}^2$$

So 1 Pa s = 10 poise (1 decapoise), and water at 20 °C, at $1.0 \times 10^{-3}$ Pa s, is 1 centipoise.

**Dimensions:**

$$[\eta] = \frac{[F]}{[A][dv/dx]} = \frac{[\mathrm{MLT^{-2}}]}{[\mathrm{L^2}][\mathrm{T^{-1}}]} = [\mathrm{ML^{-1}T^{-1}}]$$

| Fluid (20 °C) | η (Pa s) |
|---|---|
| Air | $1.8 \times 10^{-5}$ |
| Water | $1.0 \times 10^{-3}$ |
| Blood (37 °C) | about $3–4 \times 10^{-3}$ |
| Olive oil | about 0.08 |
| Glycerine | 1.4 |
| Honey | about 2–10 |

The force per unit area, $F/A$, is the **shear stress** between the layers. So Newton's law says: shear stress = $\eta \times$ velocity gradient. Compare a solid, where shear stress = modulus of rigidity × shear **strain** (Grade 11, 8.4 Moduli of elasticity): a solid resists being *twisted*, a fluid only resists being twisted *quickly*.

WORKED EXAMPLE

**Sliding a plate over oil**

A glass plate of area 0.10 m² slides at a steady 0.20 m/s over a layer of oil 1.0 mm thick on a fixed table. The oil's coefficient of viscosity is 0.80 Pa s. What force keeps the plate moving?

Answer

In a thin layer, the oil touching the table is still and the oil touching the plate moves with it, and the speed changes steadily in between. So $\frac{dv}{dx} = \frac{0.20 - 0}{1.0 \times 10^{-3}} = 200\ \mathrm{s^{-1}}$.

$$F = \eta A \frac{dv}{dx} = 0.80 \times 0.10 \times 200 = \mathbf{16\ N}$$

## 3.8 Poiseuille's formula

How much liquid flows each second through a narrow tube, a drip line, a hypodermic needle, a small blood vessel? The French doctor **Jean Léonard Marie Poiseuille** (1797–1869), who wanted to understand blood flow, measured the flow of water through glass capillaries in the 1830s and 1840s; the German engineer Gotthilf Hagen found the same rule at about the same time. The volume of liquid flowing per second, $V$, through a tube of radius $r$ and length $l$, with a pressure difference $P = P_1 - P_2$ between its ends, for a liquid of viscosity $\eta$, is

$$V = \frac{\pi P r^4}{8 \eta l}$$

This is **Poiseuille's formula**. Its assumptions:

1. The flow is steady and laminar, the streamlines parallel to the axis of the tube.
2. The pressure is the same all over any cross-section (there is no flow across the tube, only along it).
3. The liquid touching the wall is at rest (the no-slip condition).

(It also assumes the tube is straight, rigid and uniform, and the liquid incompressible and Newtonian.)



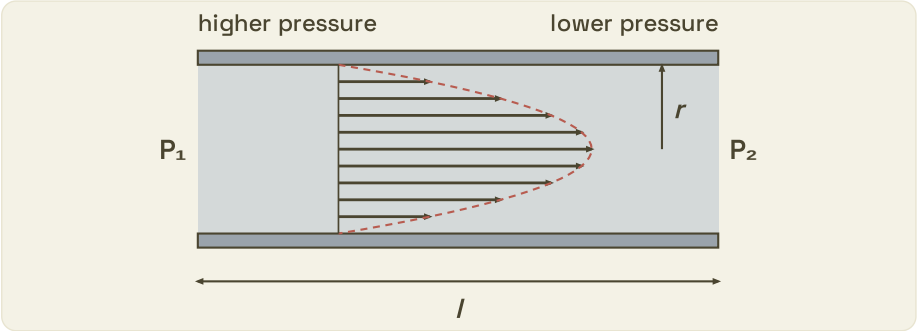



*Figure 3.23* *Liquid flowing through a tube of radius r and length l, from the higher pressure $P_1$ to the lower pressure $P_2$. The flow is fastest along the axis and still at the wall; the arrows' tips lie on a parabola.*

### Where the $r^4$ comes from: dimensions

Poiseuille's full derivation needs calculus, but the shape of the formula can be found by dimensional analysis (Grade 11, 1.6 Dimensions of a physical quantity). Suppose $V$ depends on $\eta$, $r$ and the **pressure gradient** $P/l$ (the pressure drop per unit length):

$$V = k\, \eta^a\, r^b \left(\frac{P}{l}\right)^c$$

with $k$ a number. Writing in dimensions, $[V] = [\mathrm{L^3T^{-1}}]$, $[\eta] = [\mathrm{ML^{-1}T^{-1}}]$, $[r] = [\mathrm{L}]$, $[P/l] = [\mathrm{ML^{-2}T^{-2}}]$:

$$[\mathrm{M^0L^3T^{-1}}] = [\mathrm{ML^{-1}T^{-1}}]^a\, [\mathrm{L}]^b\, [\mathrm{ML^{-2}T^{-2}}]^c$$

1. Powers of M: $0 = a + c$
2. Powers of T: $-1 = -a - 2c$
3. Powers of L: $3 = -a + b - 2c$

From 1, $a = -c$; then 2 gives $-1 = c - 2c = -c$, so $c = 1$ and $a = -1$. Then 3 gives $3 = 1 + b - 2$, so $b = 4$. Hence

$$V = k \frac{P r^4}{\eta l}$$

The full theory (or experiment) gives $k = \frac{\pi}{8}$.

The fourth power of the radius is the striking part. Doubling the pressure doubles the flow, but **doubling the radius multiplies it by 16**: a tube twice as wide has four times the area to flow through, *and* its middle is farther from the walls that hold the liquid back, so the liquid there moves four times as fast on average.

### Applications

- **Blood flow.** The body controls blood flow mainly by narrowing and widening the small arteries (arterioles): narrowing one by only 16 % halves the flow through it. A fatty deposit narrowing an artery cuts the flow sharply, and the heart must raise the pressure to make up for it; that is part of why narrowed arteries go with high blood pressure.
- **Drips and syringes.** The flow from an IV drip is set by the narrow tube and needle and by the height of the bag (the pressure); a needle of half the radius delivers a sixteenth as much at the same pressure. Thick (viscous) medicines need wider needles.
- **Lubrication.** Engineers use it to make sure oil reaches every bearing through narrow passages.
- **The laboratory.** It is used to measure viscosity (by timing a liquid through a capillary viscometer) and to design narrow tubes in instruments.

### Limitations

1. It holds only for **laminar** flow, not turbulent flow.
2. It assumes a **straight, narrow, cylindrical tube**, not a curved or irregular one.
3. It is for **incompressible Newtonian** fluids, whose viscosity doesn't change with the flow.
4. It assumes **no slip** at the walls.
5. It is for **steady** flow, so it is only a first guide for blood, which flows in pulses with each heartbeat (and isn't quite Newtonian).