Unit 3

Answers: Fluid Mechanics

Quick checks

1 c The pressure at the base is $h\rho g$: the same depth and the same liquid give the same pressure, whatever the shape. In the cone the extra water is held up partly by the sloping walls, which push back on it. This surprise is called the **hydrostatic paradox**.

2 b In the boat, the stone is floated: it displaces its own **weight** of water, a volume of m/ρ. On the bottom, it displaces only its own **volume**, m/ρ, which is smaller because the stone is denser than water. Less water displaced, so the level falls.

3 b A surface molecule is pulled inwards (the net force on it isn't zero; it is zero only deep inside). Bringing a molecule up to the surface means working against that pull, so surface molecules have more potential energy. That extra energy is the surface energy of Section 3.3, and the liquid lowers it by shrinking its surface.

4 b Splitting keeps the volume but makes much more surface (each droplet's surface is large for its small volume). More surface means more surface energy, which the sprayer had to supply. That is why atomizing a liquid takes effort.

5 a When adhesion beats cohesion, the liquid is pulled up onto the solid: it wets it, with a concave meniscus and $\theta < 90°$, like water on clean glass. A convex meniscus and $\theta > 90°$ is the opposite case, mercury on glass.

6 c $h \propto 1/r$ (Jurin's law): halving the radius doubles the rise, to 3.0 cm. Six centimeters would be for $h \propto 1/r^2$: the weight lifted grows as r^2, but the pull of the surface grows only as r, the length of the rim.

7 b A liquid's viscosity rises as it cools, so cold oil becomes thick and the engine is hard to turn over. Winter grades are made to stay thin enough in the cold (the "W" in an oil's name, such as 10W-30, is for winter). Water would freeze and doesn't lubricate.

8 d $V \propto r^4$: $(1/2)^4 = 1/16$. The area alone is a quarter, but the narrower tube also slows the liquid in it, giving another factor of four.

9 b $v \propto r^2$, so $(1/2)^2 = 1/4$. The weight (less the upthrust) grows as r^3, but the drag grows as r times v, so the speed at which they balance grows as r^2.

10 c Halving the diameter quarters the area ($A \propto d^2$), so the speed is four times as great: 12 m/s. 6 m/s would be for halving the area, not the diameter.

11 b Continuity: a smaller area means a greater speed. Bernoulli, for a horizontal pipe: where the speed is greater, the pressure is lower. The water speeds up on entering the narrow part only because the pressure behind it is higher than the pressure ahead.

12 c Mercury's surface tension, about 0.49 N/m, is nearly seven times water's (0.073 N/m): the metallic bonding between its atoms is very strong. Alcohol and oil are both far below water.

13 c A liquid expands on heating because its molecules move faster and farther apart (Grade 11, thermal expansion), whatever its surface does. The other three are all surface-tension effects.

14 **a** Surface tension is a property of the surface: only the molecules in the surface layer feel an unbalanced inward pull. A molecule in the middle is pulled equally all round.

15 **c** $h = \dfrac{2T\cos\theta}{r\rho g} \propto \dfrac{1}{g}$: a sixth of the gravity gives six times the height (if the tube is long enough).

16 **b** hr is constant (Jurin's law): $h_2 = \dfrac{h_1 r_1}{r_2} = \dfrac{7.5 \times 2}{1} = 15$ cm.

17 **d** Upthrust comes from the pressure growing with depth, $P = P_0 + h\rho g$. In free fall everything accelerates at g together and the water presses on nothing: its effective g is zero, so the pressure no longer grows with depth and there is no upthrust. (The body doesn't sink either: it is weightless too.)

18 **a** Torricelli: $v = \sqrt{2gh} = \sqrt{2 \times 10 \times 5} = 10$ m/s. Rate of flow = $av = 1 \times 10^{-4} \times 10 = 10^{-3}$ m³/s. (With g = 9.8 m/s², v = 9.9 m/s and the rate 9.9×10^{-4} m³/s, still about 10^{-3}.)

19 **d** For a floating body, the fraction under water = its density ÷ the water's density. So $\dfrac{\rho_1}{\rho_2} = \dfrac{2/3}{1/2} = \dfrac{4}{3}$.

20 **a** Doubling the diameter makes the area four times as large ($A \propto d^2$), so by continuity the speed is a quarter: 4 ÷ 4 = 1 m/s.

21 **a** $h = \dfrac{v^2}{2g} = \dfrac{2^2}{2 \times 10} = 0.20$ m = 20 cm. (With g = 9.8 m/s², 20.4 cm.) The hole's area isn't needed.

22 **c** $V = \dfrac{\pi P r^4}{8\eta l}$: the flow goes as the **fourth power** of the radius, but only as the first power of the pressure difference and inversely as the length and the viscosity. Doubling the radius gives 16 times the flow.

23 **b** The air over the top of the wing moves faster, so by Bernoulli's principle its pressure is lower than below the wing, and the pressure difference pushes the wing up.

13.

a. The **pressure** at a point in a fluid is the normal force per unit area that the fluid exerts on a surface at that point, $P = F/A$, in N/m² (Pa). At a point in a fluid at rest it acts equally in all directions. It **increases linearly with depth**: $P = P_0 + h\rho g$, so it rises by ρg for every meter deeper (about 9.8 kPa per meter in water), whatever the shape of the vessel.

b. Take a cylinder of liquid of cross-section A reaching from the surface down to depth h (see Figure 3.2).

1. Its volume is Ah, its mass $Ah\rho$, its weight $Ah\rho g$.
2. The cylinder is at rest, so the upward force on its base, PA, balances the downward forces: the air's push P_0A on its top and its weight.
3. $PA = P_0A + Ah\rho g$
4. Dividing by A: $P = P_0 + h\rho g$. The pressure due to the liquid alone (the gauge pressure) is $h\rho g$.

c. $v = \sqrt{2gh} = \sqrt{2 \times 9.8 \times 0.20} = \sqrt{3.92} = \mathbf{1.98\ m/s}$.

14.

a. A heavy bottom (ballast) lowers the ship's **center of gravity** G. A floating ship is stable when its metacenter M is above G; the lower G, the greater the metacentric height GM, and the larger the restoring couple when a wave tips the ship. A heavy keel also keeps the ship upright in a strong wind. A ship with its weight high up (a heavy deck cargo) can capsize.

b. Not usefully. A satellite and everything in it are in free fall round the Earth, so everything in it is weightless: the liquid doesn't press down on its lower parts and its pressure doesn't grow with depth. With no pressure difference there is no upthrust; the principle would give "the weight of the liquid displaced", which is zero there. A cork pushed into water in the space station stays where it is put.

c. Volume of the iron: $V = \dfrac{0.200}{8000} = 2.5 \times 10^{-5}$ m³.

Weight: $W = mg = 0.200 \times 10 = 2.00$ N.
Upthrust: $U = V\sigma g = 2.5 \times 10^{-5} \times 900 \times 10 = 0.225$ N.
Tension: $T = W - U = 2.00 - 0.225 = \mathbf{1.775\ N}$ (1.74 N if g = 9.8 m/s²).

15.

a. When a body is wholly or partly immersed in a fluid, it feels an upward force (upthrust) equal to the weight of the fluid it displaces.

b. A body fully under a liquid has two forces on it, its weight $W = V\rho g$ and the upthrust $U = V\sigma g$ (ρ the body's density, σ the liquid's).

- If $W > U$ (ρ > σ), it **sinks**.
- If $W = U$ (ρ = σ), it stays where it is put, fully under (it is suspended).
- If $W < U$ (ρ < σ), it **rises**, until only part of it is under the surface, enough that the upthrust on that part equals its weight; then it **floats**.

c. Floating: weight = upthrust, $V\rho g = 0.75V \times 1000 \times g$, so ρ = 0.75 × 1000 = **750 kg/m³**. (The volume, 0.02 m³, isn't needed.)

16.

a. In an orbiting satellite everything is in free fall and weightless, so a liquid's pressure doesn't increase with depth, and there is no upthrust: a body put in water there neither floats up nor sinks. Archimedes' principle gives the upthrust as the weight of the liquid displaced, which is zero in the satellite. (See 14b.)

b. Fraction under water $= \dfrac{\rho_{ice}}{\rho_{sea}} = \dfrac{0.92}{1.03} = 0.893$. Fraction above $= 1 - 0.893 =$ **0.107**, about a tenth: "the tip of the iceberg".

c. Let the area be A and the thickness $t = 0.25$ m. Just fully under with the man on it:

$$= +$$

$$At(1000)g = At(917)g + 80g$$

$$A \times 0.25 \times (1000 - 917) = 80 \quad \Rightarrow \quad A = \frac{80}{0.25 \times 83} = {}^{2}$$

17.

a. **Surface energy** is the extra potential energy of the molecules in a liquid's surface, compared with those inside. The surface energy per unit area is the work done to make a unit area of new surface at constant temperature.

b. Take a rectangular wire frame with a sliding wire of length l, holding a soap film (Figure 3.13). The film has two surfaces, so surface tension pulls the slider inwards with a force $F = 2Tl$. To pull the slider out slowly by x, the work done is

$$W = Fx = 2Tlx$$

The new surface made is $\Delta A = 2lx$ (both sides). So

$$W = T\,\Delta A \qquad\qquad T = \frac{W}{\Delta A}$$

The surface tension equals the surface energy per unit area (the work done per unit area of new surface).

c. **Flat:** the water meets the plate round the edge of its 6 cm × 4 cm face, a length $L = 2(6 + 4) = 20$ cm $= 0.20$ m. With $\theta = 0$ the pull is straight down:
$F = TL = 7 \times 10^{-2} \times 0.20 =$ **1.4×10^{-2} N**.
Vertical: the water meets the plate round its 6 cm × 2 mm edge, $L = 2(6 + 0.2) = 12.4$ cm $= 0.124$ m:
$F = TL = 7 \times 10^{-2} \times 0.124 =$ **8.7×10^{-3} N**.

18.

a. **Surface tension** is the force per unit length acting along a line drawn in the surface of a liquid, perpendicular to the line and along the surface: $T = F/L$. SI unit: **N/m** (the same as J/m²).

b. **Capillarity** is the rise or fall of a liquid in a narrow tube (or a narrow gap): liquids that wet the tube ($\theta < 90°$) rise, those that don't ($\theta > 90°$) are pushed down.

Derivation (Figure 3.15): let the tube have radius r, the liquid surface tension T, density ρ and angle of contact θ, and let it rise to a height h.

1. Surface tension pulls on the liquid all round the line where the meniscus meets the glass, a length $2\pi r$, at the angle θ to the wall. Its upward part is $T\cos\theta$ per unit length, so the upward force is $F = 2\pi r T\cos\theta$.
2. The column lifted (ignoring the small amount in the meniscus) has volume $\pi r^2 h$ and weight $W = \pi r^2 h\rho g$.
3. In equilibrium, $2\pi r T\cos\theta = \pi r^2 h\rho g$, so

$$h = \frac{2T\cos\theta}{r\rho g}$$

c.

$$h = \frac{2 \times 0.072 \times 1}{0.25 \times 10^{-3} \times 1000 \times 9.8} = \frac{0.144}{2.45} = 0.0588 =$$

19.

a. The **angle of contact** is the angle between the solid surface and the tangent to the liquid surface where they meet, measured through the liquid. If $\theta < 90°$ (water on clean glass, $\cos\theta$ positive), the liquid wets the tube and **rises**; the smaller θ, the higher it goes ($h \propto \cos\theta$). If $\theta > 90°$ (mercury on glass, about 140°, $\cos\theta$ negative), h is negative: the liquid is **pushed down** below the outside level. At $\theta = 90°$ (silver in water) it neither rises nor falls.

b. As in 18b: $2\pi r T\cos\theta = \pi r^2 h\rho g$, so $h = \dfrac{2T\cos\theta}{r\rho g}$, or $T = \dfrac{r\rho g h}{2\cos\theta}$.

c. Volume: $2 \times \frac{4}{3}\pi r^3 = \frac{4}{3}\pi R^3$, so $R = 2^{1/3} r$.

Surface energy before: $E_1 = 2 \times 4\pi r^2 T = 8\pi r^2 T$.
After: $E_2 = 4\pi R^2 T = 4\pi(2^{1/3} r)^2 T = 4\pi\, 2^{2/3} r^2 T$.

$$\frac{E_1}{E_2} = \frac{8}{4 \times 2^{2/3}} = \frac{2}{2^{2/3}} = 2^{1/3}$$

The ratio is $2^{1/3} : 1$, about **1.26 : 1**. The energy released heats the drop a tiny bit.

20.

a. **Viscosity** is the property of a fluid by which it opposes the relative motion between its layers. Its coefficient η has the SI unit **N s/m²** (Pa s, decapoise).

b. **Newton's law:** in laminar flow, the viscous force between two layers of fluid is directly proportional to the area A of the layers and to the velocity gradient dv/dx between them: $F = -\eta A\dfrac{dv}{dx}$, the minus sign showing that it opposes their relative motion (Section 3.7, Figure 3.22). A faster layer is dragged back by a slower one beside it, and the steeper the change of speed across the flow, the greater the drag.

c. Shear stress $= \dfrac{F}{A} = \eta\dfrac{dv}{dx} = 0.01 \times 400 =$ **4 N/m²**. (The radius isn't needed.)

21.

a. Surface tension pulls a liquid's surface as tight as it can go, so a free drop takes the shape with the **least surface area for its volume**, which is a sphere (least area means least surface energy). For a small drop, surface tension easily beats gravity, so it stays very nearly round; a large drop sitting on a table is flattened by its weight, which matters more as a drop gets bigger.

b. On a solid surface, the drop's shape depends on which is stronger:

- **cohesion > adhesion** (the liquid's molecules pull on each other more than on the solid): the drop stays rounded and doesn't wet the solid, with an angle of contact over 90°: mercury on glass, water on a lotus leaf or a waxed or oily surface;
- **adhesion > cohesion**: the drop spreads out and wets the solid, with a small angle of contact: water on clean glass, oil on almost anything.

22.

a. i. Surface energy is the extra energy of a liquid's surface; surface energy per unit area is the work done to make a unit area of new surface at constant temperature.

ii. As in 17b: pulling the slider of a soap-film frame out by x does work $W = 2Tlx$ and makes new area $\Delta A = 2lx$, so $W = T\,\Delta A$ (surface energy = surface tension × area).

b. $R = 1$ mm $= 1 \times 10^{-3}$ m; $n = 10^6$.

Equal volumes: $R^3 = 10^6 r^3$, so $r = R/100 = 1 \times 10^{-5}$ m.
Area before: $4\pi R^2 = 4\pi \times 10^{-6}$ m².
Area after: $n \times 4\pi r^2 = 10^6 \times 4\pi \times 10^{-10} = 4\pi \times 10^{-4}$ m².
Increase: $\Delta A = 4\pi(10^{-4} - 10^{-6}) = 4\pi \times 0.99 \times 10^{-4} = 1.24 \times 10^{-3}$ m².
$W = T\,\Delta A = 72 \times 10^{-3} \times 1.24 \times 10^{-3} = 8.96 \times 10^{-5}$ J $\approx$ **9×10^{-5} J**.

23.

a. **Viscosity** is the property by which a fluid opposes relative motion between its layers. The **coefficient of viscosity** η is the tangential force per unit area needed to keep a unit velocity gradient between two layers of the fluid: $\eta = \dfrac{F/A}{dv/dx}$.

b. Take layers at heights x and $x + dx$ moving at v and $v + dv$ (Figure 3.22). Experiment shows that the viscous force between them is proportional to their area, $F \propto A$, and to the velocity gradient, $F \propto \dfrac{dv}{dx}$. So $F \propto A\dfrac{dv}{dx}$, and with the constant η, $F = \eta A\dfrac{dv}{dx}$: to keep one layer moving over the other at a steady speed, this force must be applied to it, against the drag (which is $-\eta A\,dv/dx$).

c. The liquid touching the plate moves at 0.1 m/s, the liquid at the bottom is still: $\dfrac{dv}{dx} = \dfrac{0.1}{0.5 \times 10^{-3}} = 200\ \text{s}^{-1}$.

$F = \eta A\dfrac{dv}{dx} = 0.5 \times 0.2 \times 200 =$ **20 N**.

24. a. **Stokes' law:** the viscous drag on a small sphere moving slowly through a fluid is $F = 6\pi\eta r v$: directly proportional to the fluid's coefficient of viscosity η, the sphere's radius r and its velocity v. It holds for small, smooth spheres in slow, laminar flow, in a fluid that extends far round the sphere.

b. Section 3.9: at the terminal velocity, viscous drag + upthrust = weight:

$6\pi\eta r v + \frac{4}{3}\pi r^3\sigma g = \frac{4}{3}\pi r^3\rho g$, so $6\pi\eta r v = \frac{4}{3}\pi r^3(\rho - \sigma)g$, giving

$$v = \frac{2r^2(\rho - \sigma)g}{9\eta}$$

c.

$$v = \frac{2 \times (0.002)^2 \times (900 - 700) \times 9.8}{9 \times 0.1} = \frac{2 \times 4 \times 10^{-6} \times 200 \times 9.8}{0.9} = \frac{0.01568}{0.9} =$$

25. a. The volume of liquid flowing per second through a tube of radius r and length l, with a pressure difference P between its ends, for a liquid of viscosity η in steady laminar flow, is $V = \dfrac{\pi P r^4}{8\eta l}$.

b. It links the flow to the tube and the liquid, so it lets us: design pipes, drip lines and needles for a given flow; understand blood flow, which the body controls by narrowing and widening its small arteries (the r^4: a small narrowing cuts the flow a great deal); and measure a liquid's viscosity by timing its flow through a capillary. It also shows when laminar flow can be assumed, and fails when the flow turns turbulent.

c. $V = \dfrac{\pi P r^4}{8\eta l} = \dfrac{\pi \times 500 \times (2 \times 10^{-3})^4}{8 \times 0.6 \times 0.10} = \dfrac{\pi \times 500 \times 1.6 \times 10^{-11}}{0.48} = \dfrac{2.51 \times 10^{-8}}{0.48} = \times^{-83}$

26. a. Surface tension is caused by the **cohesive forces** between a liquid's molecules. A molecule inside the liquid is pulled equally in all directions; a molecule in the surface has liquid only below and beside it, so it is pulled inwards. The surface molecules therefore have extra energy, and the surface pulls itself as small as it can, like a stretched skin.
Hot soup has a **lower surface tension** than cold soup, so it spreads more easily over the tongue and reaches more taste buds; and more of its smell is carried off as vapor, and much of what we call taste is smell.

b. i. Continuity: $v_2 = \dfrac{A_1 v_1}{A_2} = \dfrac{0.02 \times 2}{0.01} = \mathbf{4\ m/s}$.

ii. Bernoulli, horizontal: $P_2 = P_1 - \frac{1}{2}\rho(v_2^2 - v_1^2) = 4 \times 10^4 - \frac{1}{2} \times 1000 \times (16 - 4) = 4 \times 10^4 - 6000 = \mathbf{3.4 \times 10^4\ Pa}$.

27.

a. For an incompressible, non-viscous fluid in steady (streamline) flow, the sum of the pressure, the kinetic energy per unit volume and the potential energy per unit volume is the same at every point along a streamline: $P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$.

b. Any two of: the lift on an aircraft's wing; sprayers and atomizers (perfume sprays, paint guns); the curving of a spinning ball (the Magnus effect); the venturimeter for measuring flow; roofs lifted off in a storm.

c. Volume flow rate: $Q = Av = \pi r^2 v = \pi \times (0.02)^2 \times 0.5 = 6.28 \times 10^{-4}$ m³/s.

The pressure difference pushes this volume through the pipe each second, and the work it does all goes into overcoming viscosity (the speed doesn't change):

$$= \Delta P \times Q = 500 \times 6.28 \times 10^{-4} =$$

The numbers given don't quite fit together, so be careful which you use. Poiseuille's formula for this pipe with 500 Pa across it would give $Q = \dfrac{\pi \times 500 \times (0.02)^4}{8 \times 0.001 \times 1} = 0.0314$ m³/s and a power of $500 \times 0.0314 = 15.7$ W; but that flow means water moving at 25 m/s, not 0.5 m/s. The speed is given, so the answer above uses it.

28.

a. To an **ideal fluid**: incompressible, non-viscous (no energy lost to friction), in steady streamline flow, along one streamline. Water and air at ordinary speeds are close enough to make it useful.

b. As in Section 3.11 (Figure 3.29): the net work done by the pressure on the fluid between two sections, $(P_1 - P_2)\Delta V$, equals the fluid's gain in kinetic and potential energy, $\frac{1}{2}\rho\Delta V(v_2^2 - v_1^2) + \rho\Delta V g(h_2 - h_1)$. Rearranging, $P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$, so $P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$.

c. $$P_2 = P_1 - \tfrac{1}{2}\rho(v_2^2 - v_1^2) = 3000 - \tfrac{1}{2} \times 1000 \times (16 - 4) = 3000 - 6000 = -3000$$

The pressure at the second point is **3000 Pa below atmospheric** (a gauge pressure of −3000 Pa; about 98 kPa absolute if the air outside is at 101 kPa). A pressure in a pipe can fall below the air's, but never below zero: that is why the 3000 Pa must be a gauge pressure.

29. a. **Equation of continuity:** for an incompressible fluid in steady flow, $A_1v_1 = A_2v_2$: the volume flowing past any cross-section per second is the same.
Derivation (Figure 3.27): in a time Δt, the fluid at section 1 moves $v_1\Delta t$, so a mass $\rho A_1 v_1 \Delta t$ enters; at section 2, a mass $\rho A_2 v_2 \Delta t$ leaves. In steady flow fluid can't pile up between them, and it can't be compressed, so the two are equal: $A_1v_1 = A_2v_2$.

b. **Bernoulli's principle:** in steady flow of an ideal fluid, where the speed is greater the pressure is lower. Mathematically, $P + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$; for a horizontal pipe, $P + \frac{1}{2}\rho v^2 = \text{constant}$.

c. For fluid to speed up, a net force must push it forward, and in a fluid that force comes from the pressure: the pressure behind it must be higher than the pressure ahead. So fluid speeds up only while moving from high pressure into low, and where it is fastest the pressure is lowest. In energy terms, the total energy per unit volume, $P + \frac{1}{2}\rho v^2$, stays the same, so more kinetic energy means less pressure energy.

d. i. $v_2 = v_1\dfrac{A_1}{A_2} = v_1\left(\dfrac{d_1}{d_2}\right)^2 = 2 \times \left(\dfrac{10}{5}\right)^2 = \mathbf{8\ m/s}$.

ii. $P_1 - P_2 = \frac{1}{2}\rho(v_2^2 - v_1^2) = \frac{1}{2} \times 1000 \times (64 - 4) = \mathbf{3.0 \times 10^4\ Pa}$, higher at the wide end.