Unit 2

Periodic Motion

This unit online, with its interactives:
science-playground.com/s/12f2

Push a child on a swing and let go: it swings out, back, out again, each swing taking the same time. A guitar string hums at one pitch, a wall clock's pendulum ticks off the seconds, the needle of a sewing machine goes up and down, your heart beats, and the Earth goes round the Sun once a year. Motions that repeat themselves are everywhere.

The simplest and most important of them is **simple harmonic motion**, the motion of a mass on a spring or a pendulum making small swings. It is worth learning well: waves, sound, alternating current, light and even the vibrations of atoms in a solid are all built from it. This unit describes it with one equation, finds its energy, works out the periods of a spring, a pendulum and a twisting disc, and ends with what happens when friction slows an oscillation down, or a push keeps it going: damping and resonance.

2.1 Introduction

Periodic motion is a motion that repeats itself at regular intervals of time. The time for one repeat is its **period**. The swing of a pendulum, the ticking of a clock, the vibration of a guitar string, the turning of a fan and the Moon going round the Earth are all periodic.

A periodic motion in which a body moves **back and forth about a fixed point**, along the same path, is an **oscillation** (or vibration). Every oscillation is periodic, but not every periodic motion is an oscillation: a fan's blade goes round, not back and forth.

Three questions to keep in mind:

1. Why does a swinging pendulum always come back? (Section 2.2: a restoring force.)
2. How do musical instruments make their notes? (Natural frequencies and resonance, Section 2.7.)
3. Can a motion repeat forever without stopping? (Damping, Section 2.7.)

2.2 Simple harmonic motion

A mass on a spring, a pendulum making small swings, the prong of a tuning fork and a child on a swing all oscillate about a position of rest, the **mean position** (or equilibrium position). Pull any of them aside and a force pulls it back: a **restoring force**. In each, the restoring force grows in step with how far the body is from the mean position: twice as far, twice the pull. Motion of this kind is **simple harmonic motion (SHM)**.

Simple harmonic motion is a periodic motion in which the restoring force is directly proportional to the displacement from the mean position and is always directed towards the mean position.

$$F = -kx$$

where x is the displacement from the mean position and k is a constant (for a spring, its spring constant). The minus sign says that the force points the opposite way to the displacement: towards the mean position.

By Newton's second law, $F = ma$, so

$$ma = -kx \qquad \Rightarrow \qquad a = -\frac{k}{m}x$$

k/m is a positive constant; it is written ω^2 (Section 2.2.1 shows why), so

$$a = -\omega^2 x$$

So SHM can also be defined by its acceleration: **SHM is an oscillation in which the acceleration is directly proportional to the displacement from the mean position and is always directed towards the mean position.** Whenever we find that a system obeys $F = -kx$, or $a = -(\text{a constant})\,x$, we know at once that it moves in SHM, and its ω is the square root of that constant.

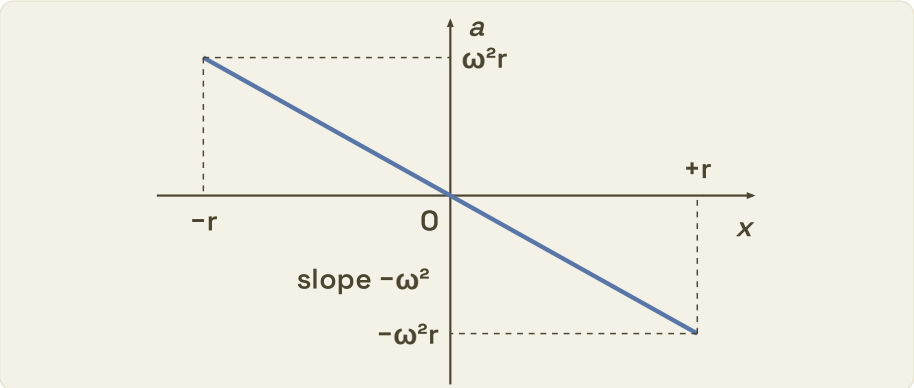


Figure 2.1 *The acceleration–displacement graph of SHM: a straight line through the origin with a negative slope, $-\omega^2$. On the right of the mean position (x positive) the acceleration points left; on the left it points right.*

Check 1

Which of these is simple harmonic motion?

a a ball bouncing on the floor
b a car going round a circular track at steady speed
c a mass bobbing up and down on a spring
d a falling stone

2.2.1 The equation of SHM and its terms

There is a neat link between SHM and steady circular motion. Let a point P go round a circle of radius r, centered at O, with a steady angular velocity ω, and watch its **shadow** on a vertical diameter (its projection, N). As P goes round, N moves up and down the diameter, between $+r$ and $-r$, and its motion is exactly SHM. The circle is called the **reference circle**.

Let P start at X, on the horizontal line through O, at $t = 0$, and go round anticlockwise (to Y, X′, Y′ and back to X). After a time t it has turned through an angle $\theta = \omega t$.

(i) Displacement. The displacement y of the oscillating particle is its distance from the mean position at any instant. N is at height $r\sin\theta$ above O, so

$$y = r\sin\omega t \tag{2.1}$$

where y is the displacement at time t, r the amplitude and ω the **angular frequency**. The displacement follows a sine curve in time.

(ii) Velocity. Differentiating,

$$v = \frac{dy}{dt} = r\omega\cos\omega t$$

Since $\cos\omega t = \sqrt{1-\sin^2\omega t} = \sqrt{1-y^2/r^2}$ (taking the size),

$$v = r\omega\sqrt{1-\frac{y^2}{r^2}} \qquad\Rightarrow\qquad v = \omega\sqrt{r^2-y^2} \tag{2.2}$$

- At the mean position, $y = 0$: $v = r\omega = v_{\max}$, the **greatest** (maximum) speed.
- At an extreme position, $y = \pm r$: $v = 0$. The particle stops for an instant and turns back.

(iii) Acceleration. Differentiating again,

$$a = \frac{dv}{dt} = \frac{d}{dt}(r\omega\cos\omega t) = -r\omega^2\sin\omega t \qquad\Rightarrow\qquad a = -\omega^2 y \tag{2.3}$$

So the acceleration is directly proportional to the displacement and directed towards the mean position: the motion of N is SHM, with the same ω as P's angular velocity. (It is also the vertical part of P's centripetal acceleration, $r\omega^2$, which points to the center.)

- At the mean position, $y = 0$: $a = 0$.
- At an extreme position, $y = \pm r$: $a = \mp\omega^2 r$, the **greatest** size of acceleration, $a_{\max} = \omega^2 r$.

The speed is greatest where the acceleration is zero, and the acceleration greatest where the particle stops.

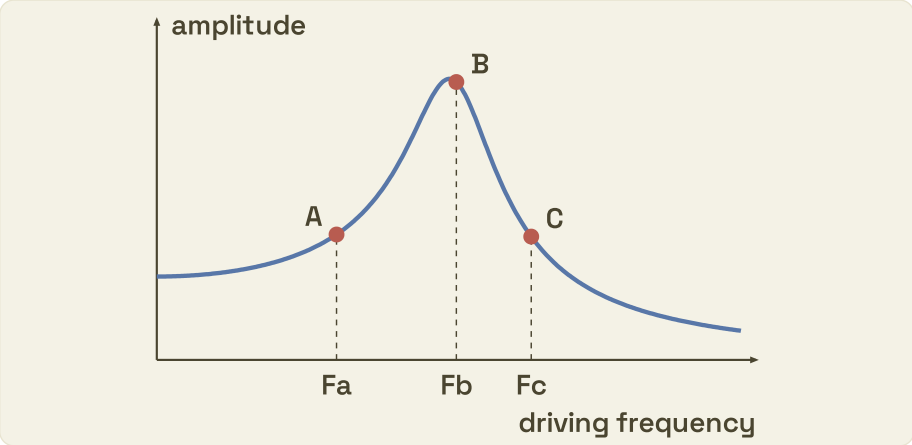


Figure 2.15 *For question 20: amplitude against the frequency of the periodic force; A, B and C are at frequencies Fa, Fb and Fc.*

23. A system oscillates with a starting amplitude of 0.5 m. Because of damping, its amplitude falls by 5% each cycle.

i. Find the amplitude after 10 oscillations.

ii. If its starting energy is 40 J, how much energy is left after 10 oscillations?

A companion to your textbook, not a replacement for it: study from the book your school uses, and where the two differ, follow your teacher. Found a mistake? Please tell us.

(iv) Amplitude. The **amplitude** is the greatest displacement from the mean position. Since $\sin\omega t$ is at most 1, $y_{\max} = r$: the amplitude is the radius of the reference circle. (It is sometimes called the displacement amplitude.)

(v) Period. The **period**, T, is the time for one complete oscillation: P goes once round its circle, through 2π radians, so $\omega T = 2\pi$ and

$$T = \frac{2\pi}{\omega}$$

From $a = -\omega^2 y$, taking sizes, $\omega = \sqrt{\dfrac{|a|}{|y|}}$, so

$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

This is a general recipe: for any SHM, find the acceleration at a displacement, and the period follows. It works because the ratio of the two is the same everywhere in an SHM.

(vi) Frequency. The **frequency**, f, is the number of complete oscillations in one second, the reciprocal of the period:

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{\text{acceleration}}{\text{displacement}}}, \qquad \omega = 2\pi f$$

Its unit is the **hertz** (Hz), one oscillation per second. The angular frequency ω is in rad/s.

(vii) Phase. The **phase** of an oscillating particle tells where it is in its cycle, and which way it is moving, at an instant. If the particle is at the mean position at $t = 0$, its displacement is $y = r\sin\omega t$; in general

$$y = r\sin(\omega t + \phi)$$

and the angle $\theta = \omega t + \phi$ is its phase. ϕ is the phase at $t = 0$, the **initial phase** or **epoch**: it says where in the cycle the particle starts. With $\phi = 0$, it starts at the mean position, moving up; with $\phi = \pi/2$ (90°), at its upper extreme, so then $y = r\cos\omega t$. The phase changes steadily with time; the epoch stays the same.

Phase difference

Two oscillations of the same frequency can be at different points of their cycles. The **phase difference** between them is the difference of their phases, in degrees or radians: how far one is ahead of (leads) or behind (lags) the other.

- **In phase** (phase difference 0, or 2π, 4π…): they reach their extremes together and pass the mean position together, moving the same way.
- **In antiphase** (completely out of phase, phase difference π or 180°): when one is at its upper extreme, the other is at its lower extreme. Two waves of the same amplitude meeting in antiphase cancel each other; in phase, they add.

Phase differences matter wherever oscillations are combined: waves interfering, the voltage and current in an AC circuit, or signals in communication.

WORKED EXAMPLE

A particle in SHM

A particle moves in SHM with a period of 2 s and an amplitude of 6 cm. At $t = 0$ it is at its greatest displacement and starts moving towards the mean position.

a. Find its angular frequency.

b. Find its velocity and acceleration when its displacement is 4 cm.

c. Find its greatest speed and greatest acceleration.

Answer

a. $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{2} = \boldsymbol{\pi}$ **rad/s** (3.14 rad/s).

b. It starts at an extreme, so its phase at $t = 0$ is $\pi/2$ and $y = 6\cos\pi t$ cm; but the speed at a displacement doesn't depend on where it started:

$v = \omega\sqrt{r^2 - y^2} = \pi\sqrt{6^2 - 4^2} = \pi\sqrt{20} = 2\sqrt{5}\,\pi =$ **14.0 cm/s** (towards or away from the mean position, depending on the instant).
$a = -\omega^2 y = -\pi^2 \times 4 = -4\pi^2 =$ **−39.5 cm/s²**: 39.5 cm/s² towards the mean position.
c. Greatest speed, at the mean position: $v_{\max} = r\omega = 6\pi =$ **18.8 cm/s**.
Greatest acceleration, at the extremes: $a_{\max} = \omega^2 r = 6\pi^2 =$ **59.2 cm/s²**.

YOUR TURN

Mean position to the extreme

A particle oscillates as $y = 7.0\sin 0.5\pi t$, with y in mm and t in seconds. How long after the motion begins does it first move from the mean position to its greatest displacement?

Answer

At $t = 0$, $y = 0$: it starts at the mean position. It first reaches $y = 7.0$ mm when $\sin 0.5\pi t = 1$, that is when $0.5\pi t = \dfrac{\pi}{2}$, so $t = \mathbf{1\ s}$. (Check: $\omega = 0.5\pi$, so $T = 2\pi/\omega = 4$ s, and mean to extreme is a quarter of a period.)

Check 2

A particle in SHM is at its greatest displacement. At that instant:

a its speed and acceleration are both greatest
b its speed is zero and its acceleration is greatest
c its speed is greatest and its acceleration is zero
d its speed and acceleration are both zero

2.3 Energy in SHM

As a particle oscillates, its energy keeps changing from kinetic to potential and back. With no friction, the total stays the same.

Potential energy. Let a particle of mass m be at displacement y. Its acceleration is $a = -\omega^2 y$, so the restoring force is

$$F = ma = -m\omega^2 y = -ky, \qquad k = m\omega^2$$

the form of Hooke's law. To move the particle a further small step dy away from the mean position, we must push against this force with an equal and opposite force ky, doing work $dW = ky\,dy$. The work done to take it from the mean position to y is stored as potential energy:

$$PE = W = \int_0^y ky\,dy = \tfrac{1}{2}ky^2 = \tfrac{1}{2}m\omega^2 y^2$$

Kinetic energy. From Equation 2.2, $v^2 = \omega^2(r^2 - y^2)$, so

$$KE = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\omega^2(r^2 - y^2)$$

Total energy.

$$E = KE + PE = \tfrac{1}{2}m\omega^2(r^2 - y^2) + \tfrac{1}{2}m\omega^2 y^2 \qquad \Rightarrow \qquad \boxed{E = \tfrac{1}{2}m\omega^2 r^2}$$

with $\omega = \dfrac{2\pi}{T} = 2\pi f$. The y has canceled: the total energy is the same at every point of the motion. It is proportional to the **square of the amplitude** (and of the frequency): double the amplitude and the energy is four times as much.

- **At the mean position** ($y = 0$): $PE = 0$, and the kinetic energy is greatest, $KE = \tfrac{1}{2}m\omega^2 r^2 = E$. All the energy is kinetic.
- **At an extreme** ($y = \pm r$): $KE = 0$ and $PE = \tfrac{1}{2}m\omega^2 r^2 = E$. All the energy is potential.

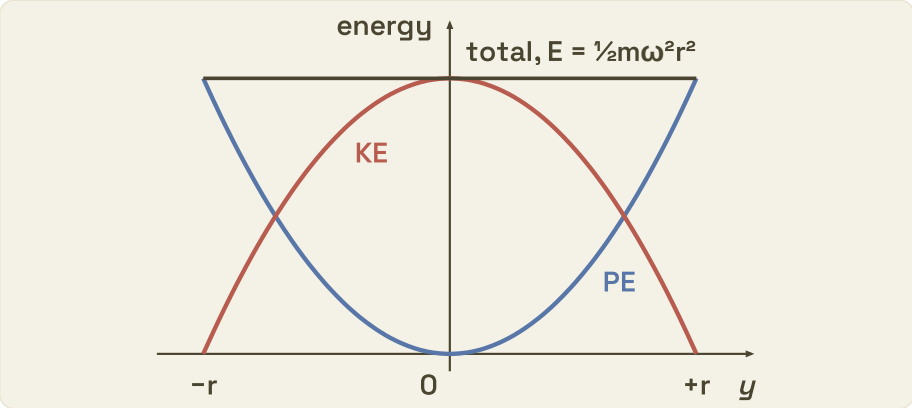


Figure 2.2 *Energy against displacement in SHM. The potential energy ½mω²y² is a parabola, zero at the mean position; the kinetic energy ½mω²(r² − y²) is the upside-down parabola, greatest there; their sum, the total energy ½mω²r², is the same everywhere.*

16. The diagram shows a mass hung from a helical spring, oscillating up and down. A motion sensor connected to a data logger records its motion.
    a. Name the type of oscillation shown.
    b. Write the expression for the period T of the vertical oscillation in terms of the mass m and the spring constant k.
    c. If the extension of the spring at equilibrium is h, derive a relation between h, g and T.

17. A torsion pendulum moves in angular SHM.
    a. What is angular simple harmonic motion? Give one example.
    b. Derive the expression for the period of angular SHM of a torsion pendulum.
    c. A disc of moment of inertia 0.05 kg m² hangs from a wire of torsional constant 0.10 N m/rad. Find the period of its angular oscillation.

18. A pendulum in air meets air resistance and slows down over time.
    a. What is damping in an oscillation? Name the types of damping.
    b. Explain what happens to the amplitude when the frequency of a periodic driving force matches the natural frequency of the system.
    c. A metal pendulum bob oscillates in still air. An investigator records its successive peak displacements on the same side of its equilibrium position, one period apart:

Peak number	1	2	3	4
Displacement (cm)	8.0	6.0	4.5	3.4

    Assume the oscillations are small.
    i. Describe how you would sketch a displacement–time graph for this pendulum from these data. How does the graph show that the bob is damped?
    ii. If the undamped period of the pendulum is 1.6 s, will its actual period in air be exactly 1.6 s, slightly more, or slightly less? Give a reason.
    iii. The pendulum is now made to swing in thin oil instead of air. State two changes you would expect in the displacement–time graph.

19. A tuning fork vibrates under an outside periodic force.
    a. What is resonance? Under what condition does it occur?
    b. What is the phase relation between the displacement and the driving force at resonance?
    c. A forced oscillator has a natural (angular) frequency $\omega_0 = 5$ rad/s. When it is driven by an outside force of angular frequency $\omega = 5$ rad/s, its amplitude is greatest. Explain this with a graph, and state the resonance condition with numbers.

20. The graph shows how the amplitude of a system varies with the frequency of a periodic force applied to it. Points A, B and C are at the frequencies Fa, Fb and Fc.
    a. Name the physical phenomenon the graph shows.
    b. Identify the frequency Fb, and explain what happens to the amplitude at it.
    c. Compare the amplitudes at A and C, and explain why they are lower than at B.

21. A block of mass m attached to a vertical spring moves in SHM.
    a. Define SHM, and derive its equation of motion using Hooke's law.
    b. What is the total mechanical energy in SHM? Derive the expression, and discuss how the energy varies with time.
    c. Explain why the restoring force in vertical SHM does not depend on gravity.
    d. Two springs, of spring constants $k_1 = 200$ N/m and $k_2 = 300$ N/m, are connected in parallel to a mass of 2 kg.
    i. Find the effective spring constant.
    ii. Calculate the period of oscillation.
    iii. If the amplitude is 0.15 m, find the energy of the oscillation (the greatest energy stored in the springs).

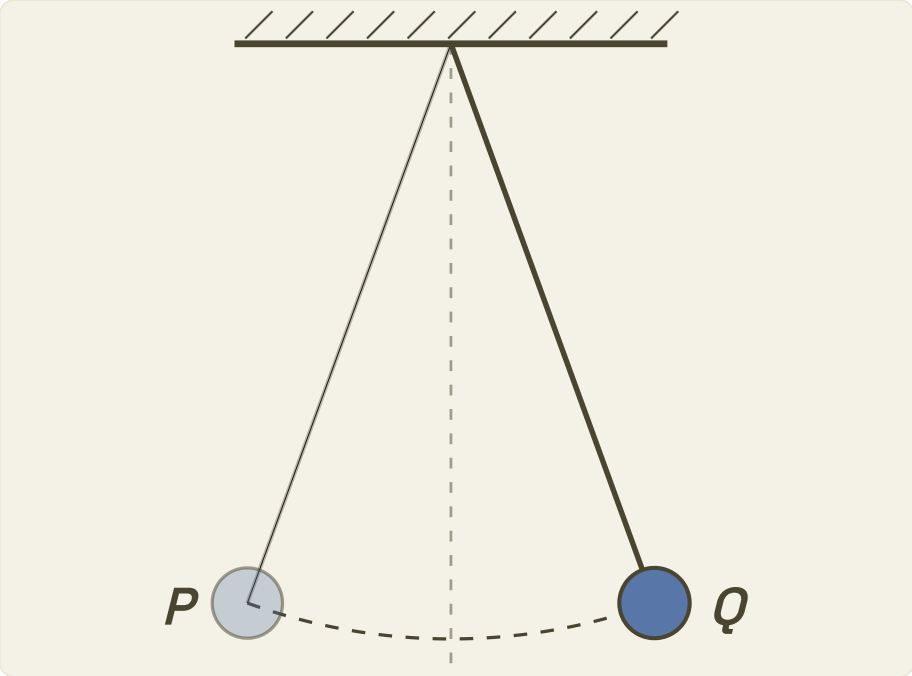


Figure 2.13 For question 14: a simple pendulum swinging between P and Q.

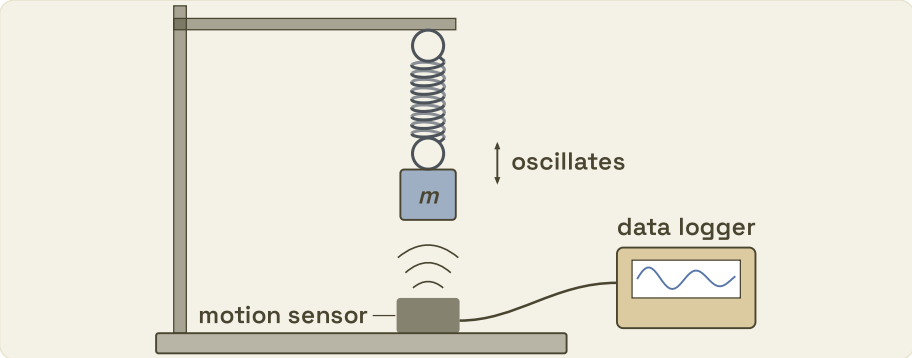


Figure 2.14 For question 16: a mass oscillating up and down on a helical spring, with a motion sensor on the floor below it, connected to a data logger.

Check 3
At what displacement are the kinetic and potential energies of a particle in SHM equal?

a $y = r/2$ b $y = r/\sqrt{2}$
c $y = r/4$ d $y = r$

2.4 Applications of SHM

Simple harmonic motion is everywhere in engineering. A car's springs and shock absorbers, a building's sway, the quartz crystal in a watch, a guitar string and the balance wheel of a mechanical watch all oscillate in, or close to, SHM, and engineers design them from its equations. Here is one of the most useful: a mass hanging from a spring, as in a vehicle's suspension.

2.4.1 Vertical oscillations of a spring–mass system

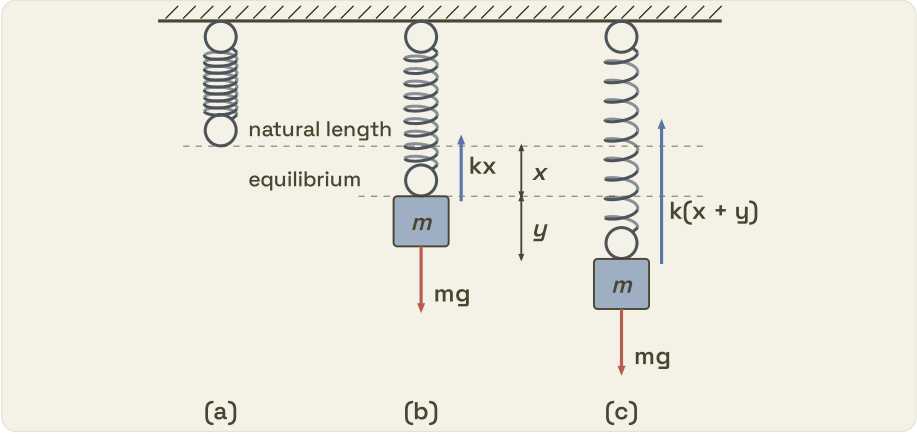


Figure 2.3 A mass on a vertical spring. (a) The spring, unstretched. (b) The mass m hangs at rest, the spring stretched by x, where $kx = mg$: its equilibrium (mean) position. (c) Pulled down a further y, the spring pulls up with $k(x + y)$, more than the weight; let go, the mass oscillates about the equilibrium position.

A light spring of spring constant k hangs from a fixed support, and a mass m is hung from it.

1. **Equilibrium.** The mass stretches the spring by x and comes to rest. The spring pulls up with kx (Hooke's law) and the weight pulls down with mg; at rest the net force is zero, so
$$kx = mg$$
2. **Displaced.** Pull the mass down a further distance y and let go. The spring is now stretched by $x + y$, so it pulls up with $k(x + y)$, while the weight still pulls down with mg. Taking downward as positive, the net force on the mass is
$$F = mg - k(x + y) = mg - kx - ky = -ky$$
since $kx = mg$. The same holds above the equilibrium position (y negative).
3. **SHM.** By Newton's second law, $ma = -ky$, so
$$a = -\frac{k}{m}y = -\omega^2 y, \qquad \omega = \sqrt{\frac{k}{m}} \tag{2.4}$$
The acceleration is proportional to the displacement from the equilibrium position and points towards it: the mass moves in SHM about the **equilibrium** position (not the unstretched length).
4. **Period.**
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{2.5}$$
So the period of a mass on a spring depends only on the **mass hung from it** and the **spring constant**, not on the amplitude and not on g. Gravity only moves the point it oscillates about, down by $x = mg/k$; the same mass and spring would oscillate with the same period on the Moon, or lying on a smooth table. (We have taken the spring's own mass to be negligible.) A heavier mass oscillates more slowly; a stiffer spring, faster.

Since $m/k = x/g$, the period can also be found from the stretch alone: $T = 2\pi\sqrt{\frac{x}{g}}$.

Springs joined together

Two springs, of spring constants k_1 and k_2, can hold a mass together in two ways (Figure 2.4).

- **In parallel** (side by side, sharing the load): pulled down by y, both stretch by y, and their pulls add: $F = -(k_1 + k_2)y$. So $k = k_1 + k_2$. The pair is stiffer than either.
- **In series** (one below the other): the same force F passes through both, so they stretch by F/k_1 and F/k_2, and the total stretch is $F\left(\frac{1}{k_1} + \frac{1}{k_2}\right)$. So $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$. The pair is softer than either: two equal springs in series have $k/2$.

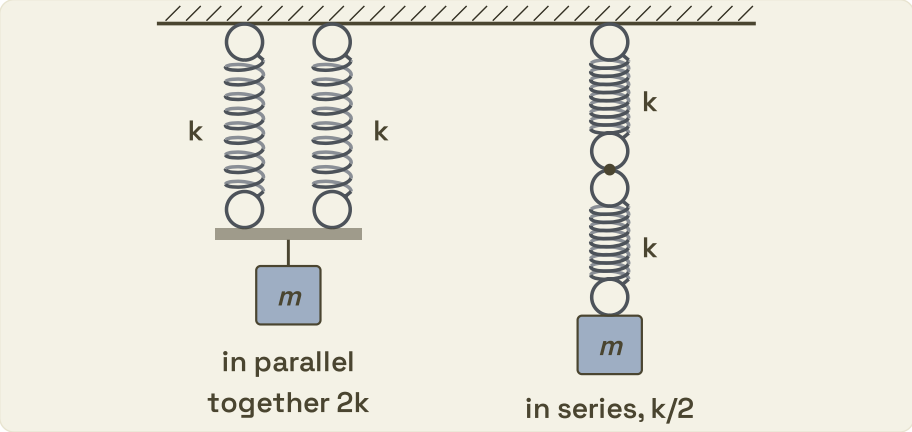


Figure 2.4 Two equal springs, each of spring constant k, holding a mass m: in parallel the pair has 2k; in series, k/2.

Check 4

A mass on one spring oscillates with period T. The same mass is hung from two such springs in parallel. Its period is now:

a 2T
b $\sqrt{2}T$
c $T/\sqrt{2}$
d T/2

2.5 Simple pendulum

The first recorded study of the pendulum was by Galileo Galilei, at the end of the 1500s. The story goes that, as a young man in the cathedral of Pisa, he watched a lamp swinging on its long chain and timed its swings against his pulse: as the swings grew smaller, each still seemed to take the same time. That a pendulum's period hardly depends on its amplitude (for small swings) is called **isochronism**. In 1656–57 the Dutch scientist Christiaan Huygens used it to build the first pendulum clock, which kept time to within about a minute a day, against about a quarter of an hour for the best clocks before it.

A **simple pendulum** is a small, heavy bob hung from a fixed point by a light, inextensible string, swinging to and fro in a vertical plane under gravity. For small swings its motion is SHM.

- The **point of suspension** is the fixed point from which the pendulum hangs.
- The **point of oscillation** is the bob's center of gravity.
- The **effective length**, l, is the distance from the point of suspension to the point of oscillation: the length of the string plus the radius of the bob.

The ideal simple pendulum assumes a point-sized bob, a massless string that doesn't stretch, no friction or air resistance, and small swings.

Check 15

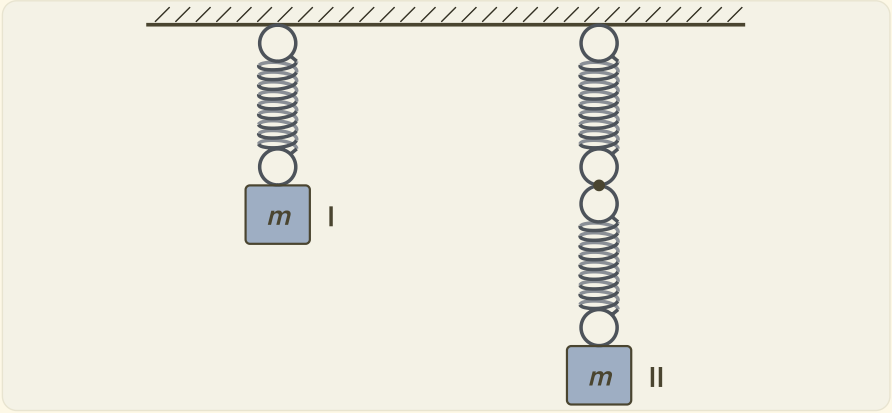


Figure 2.12 For question 9: (I) a mass on one spring; (II) the same mass on two such springs, one below the other.

9. In the picture above, the natural frequency of system I is f. What is the frequency of system II, the same mass on two similar springs joined end to end?

a f
b $f/\sqrt{2}$
c $\sqrt{2}/f$
d $\sqrt{2}f$

Check 16

10. When does resonance occur in a system that can oscillate?

a when it is disturbed by a random force
b when the driving frequency matches its natural frequency
c when its amplitude is reduced
d when it loses energy

Check 17

11. Where would you see resonance in everyday life?

a in a pendulum swinging with no outside force
b in a car's suspension absorbing a shock
c in musical instruments such as a guitar or violin
d in a spring stretched by a steady weight

Check 18

12. In an angular simple harmonic oscillator the restoring torque is $\tau = -k\theta$, where k is the torsional constant. If I is its moment of inertia, its angular frequency is:

a $\sqrt{k}$
b $\sqrt{k/I}$
c k/I
d I/k

Exercise

Choose the correct option

Check 7

1. A particle's displacement in SHM is $y = 3\sin\omega t + 4\cos\omega t$, in cm. Its amplitude is:

a 3 cm
b 4 cm
c 5 cm
d 7 cm

Check 8

2. What is the angle between the instantaneous velocity and the acceleration of a particle in SHM?

a zero
b $\pi/2$
c π
d zero or π

Check 9

3. A particle in SHM has amplitude r and period T. How long does it take to go from x = r to x = r/2?

a T/2
b T/3
c T/6
d T/12

Check 10

4. What happens to the period of a simple pendulum if its length is tripled?

a it doubles
b it triples
c it increases by a factor of √3
d it increases by a factor of 3

Check 11

5. A pendulum has a period of 2 s on the Earth. What is its period on the Moon, where g is a sixth of the Earth's?

a about 5 s
b 2 s
c 6 s
d 12 s

Check 12

6. A girl on a swing swings with period T. Another girl of the same size sits beside her, and they swing together. The period:

a is halved
b is doubled
c stays the same
d is four times as long

Check 13

7. The mass on a spring is halved and the spring's stiffness doubled. Its natural frequency:

a is halved
b is doubled
c is four times as much
d is unchanged

Check 14

8. What happens to the frequency of a mass–spring system if the mass is doubled?

a it doubles
b it stays the same
c it decreases by a factor of √2
d it decreases by a factor of 2

Answer these questions

13. A body moves in simple harmonic motion along a straight line.

a. Define simple harmonic motion. What are its essential characteristics?
b. Derive the expression for the acceleration in SHM.
c. A particle of mass 0.1 kg moves in SHM with a period of 2 s and an amplitude of 0.5 m.
i. Find its total mechanical energy.
ii. Find its kinetic energy when its displacement is 0.3 m.
iii. At what displacement is its speed half its greatest value?

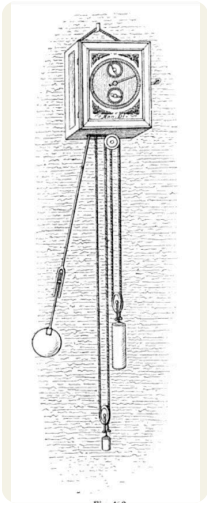


Figure 2.5 *Huygens' first pendulum clock (1657), in a drawing made in 1899. The pendulum swinging below the case set the clock's pace; the hanging weights drove it.*
Photo: Unknown author, Public domain

The period of a simple pendulum

Let the bob, of mass m, be pulled aside and let go. At some instant it is at B, at an angle θ to the vertical OA, a distance x along the arc from its mean position A.

1. **The forces.** Two forces act on the bob: its weight mg, straight down, and the tension T in the string, along the string towards O. Split the weight into two parts:
   - **along the string**, $mg\cos\theta$, away from O. The tension balances it (and, since the bob moves on a circle, the tension is a little larger, to give the centripetal force);
   - **along the arc** (perpendicular to the string), $mg\sin\theta$, towards A. Nothing balances it: it is the **restoring force**.
2. **The restoring force** acts opposite to the displacement, so
$$F = -mg\sin\theta$$
3. **Small angles.** For a small angle, in radians, $\sin\theta \approx \theta$ (at 5°, they differ by only 0.1 %), and the arc gives $\theta = \dfrac{x}{l}$. So
$$F = -mg\theta = -\frac{mg}{l}x$$
4. **SHM.** With $F = ma$,
$$a = -\frac{g}{l}x = -\omega^2 x, \qquad \omega = \sqrt{\frac{g}{l}} \tag{2.6}$$
The acceleration is proportional to the displacement and directed towards the mean position: for small swings, the pendulum moves in SHM.
5. **Period.**
$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}} \tag{2.7}$$

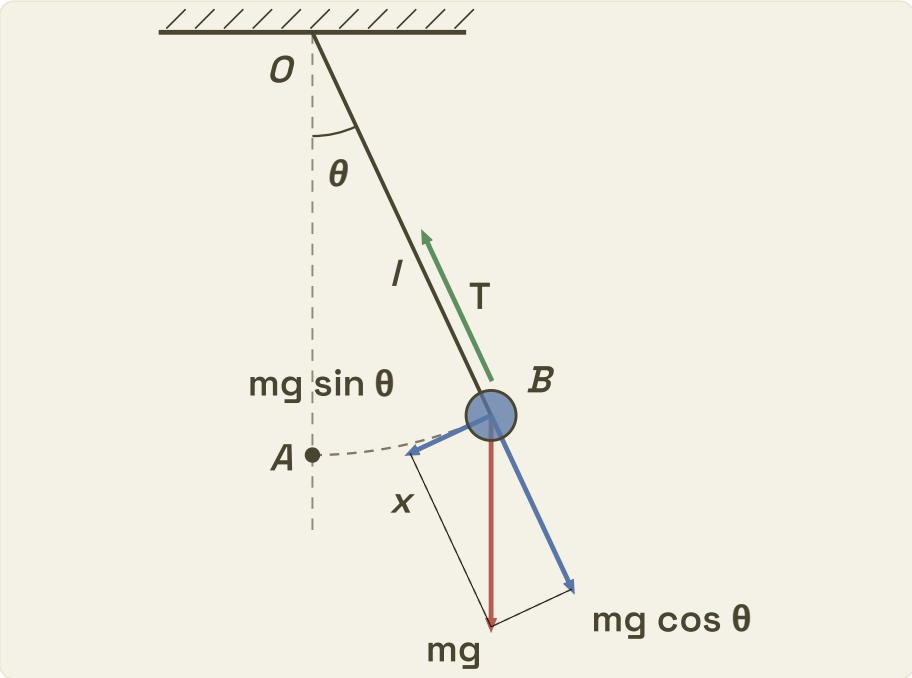


Figure 2.6 *A simple pendulum displaced through an angle θ. Its weight mg is split along the string (mg cos θ, balanced by the tension T) and along the arc (mg sin θ, the restoring force towards A). The bob is a distance x along the arc from its mean position A.*

The l–T^2 graph and measuring g

Squaring Equation 2.7,

$$T^2 = \frac{4\pi^2}{g}\,l$$

This has the form $y = mx$: a graph of T^2 against l is a **straight line through the origin**, with slope $\frac{4\pi^2}{g}$. Time a pendulum at several lengths, plot T^2 against l, and measure the slope: then $g = \frac{4\pi^2}{\text{slope}}$. Using the slope of a best-fit line averages out the errors in single readings, so it gives g more reliably than one length alone.



Figure 2.11 *The Tacoma Narrows Bridge on 7 November 1940, as the middle of its deck, after an hour of violent twisting in the wind, broke away and fell into the water.*
Photo: Wikimedia Commons, Public domain

DID YOU KNOW?

Your legs are pendulums

When you walk, each leg swings forward from the hip almost freely, like a pendulum. A longer pendulum swings more slowly, so a tall adult, with long legs, naturally takes slower steps than a small child, who seems to trot to keep up. Walking at the pace at which the legs swing by themselves takes the least effort, which is why each of us has a natural, comfortable walking pace.

IN SHORT

Key ideas of Unit 2

- **Periodic motion** repeats at regular intervals; an **oscillation** is a periodic to-and-fro motion about a mean position.
- **SHM:** restoring force $F = -kx$, or acceleration $a = -\omega^2 x$, proportional to the displacement and towards the mean position.
- $y = r\sin(\omega t + \phi)$; $v = \omega\sqrt{r^2 - y^2}$ (greatest, $r\omega$, at the mean position); $a = -\omega^2 y$ (greatest, $\omega^2 r$, at the extremes).
- $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$, $f = \frac{1}{T}$, $\omega = 2\pi f$. Phase $\omega t + \phi$; ϕ is the epoch.
- Energy: $PE = \frac{1}{2}m\omega^2 y^2$, $KE = \frac{1}{2}m\omega^2(r^2 - y^2)$, total $E = \frac{1}{2}m\omega^2 r^2$, constant.
- Mass on a spring: $T = 2\pi\sqrt{m/k}$, independent of g. Springs in parallel $k_1 + k_2$; in series $1/k = 1/k_1 + 1/k_2$.
- Simple pendulum (small swings): $T = 2\pi\sqrt{l/g}$; $T^2 \propto l$; a seconds pendulum (T = 2 s) is 0.993 m long.
- Torsion pendulum: $\tau = -\kappa\theta$, $T = 2\pi\sqrt{I/\kappa}$.
- Damping (light, critical, heavy) takes energy away; a forced oscillation is at the driving frequency; at **resonance** (driving frequency = natural frequency) the amplitude is greatest and the displacement lags the force by $\pi/2$.

With light damping, the amplitude at resonance can be very large; heavier damping keeps it smaller and makes the peak broader.

The phase also changes as the driving frequency passes resonance. Driven slowly, the body moves in step with the force. Driven fast, it moves opposite to the force (half a cycle behind). **At resonance, the displacement lags the driving force by a quarter of a cycle, π/2 (90°)**: the body's velocity is then in step with the force, so the force pushes the way it moves all the time, putting in energy as fast as it can.

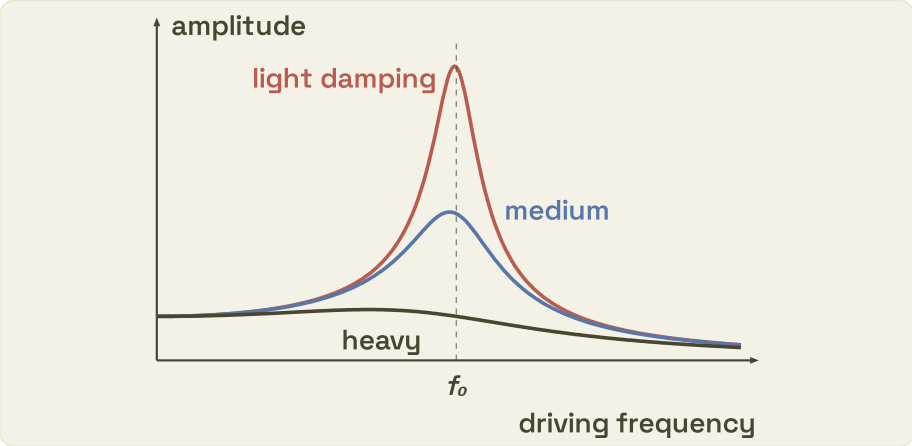


Figure 2.10 *Amplitude against driving frequency for a forced oscillation, for light, medium and heavy damping. Each curve peaks near the natural frequency f_o; the lighter the damping, the higher and sharper the peak.*

Examples of resonance:

1. **A swing pushed at the right time:** pushes at the swing's natural frequency make it go higher and higher.
2. **A tuning fork on a resonance box:** a wooden box whose air vibrates at the fork's frequency makes its sound much louder. The hollow body of a guitar, violin or sarangi does the same for the strings.
3. **Breaking a glass with sound:** a loud note at a wine glass's natural frequency can set it vibrating so strongly that it shatters. (It takes a very loud, steady note; it has been done with a loudspeaker and, rarely, by a trained voice.)
4. **Tuning a radio:** a radio's tuning circuit has a natural frequency that you change by turning the knob; when it matches a station's frequency, that station's signal makes it resonate, and the radio picks it out from all the others.
5. **Buildings in an earthquake:** the ground shakes most at certain frequencies; a building whose natural frequency matches them sways far more than its neighbors. Engineers design buildings to avoid this, and tall buildings carry tuned mass dampers, huge weights on springs that soak up the sway.
6. **Marching across a bridge:** soldiers are ordered to break step on a bridge, so that their steps don't drive it at its natural frequency.

CAREFUL

The Tacoma Narrows Bridge

Some books say the Tacoma Narrows Bridge in the USA collapsed in 1940 because gusts of wind matched its natural frequency. The wind that day was steady, about 64 km/h. The bridge's twisting itself changed how the wind pushed on it, so that each twist drew more energy from the wind (aeroelastic flutter, a self-excited oscillation). It is still a lesson about letting a structure oscillate at its natural frequency, and in an exam it is usually accepted as an example of resonance.

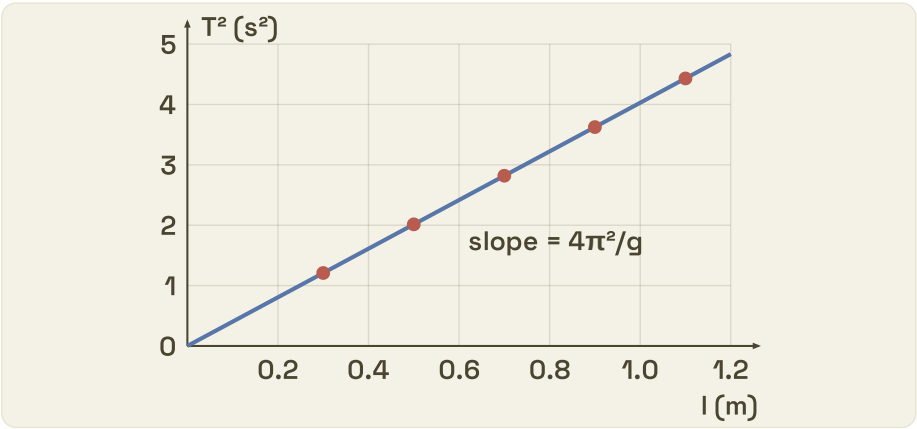


Figure 2.7 *T^2 against l for a simple pendulum where $g = 9.8$ m/s²: a straight line through the origin with slope $4\pi^2/g = 4.03$ s²/m.*

The seconds pendulum

A **seconds pendulum** is a simple pendulum whose period is **2 s**: each swing, one way, takes 1 s, so it "beats seconds", as in a grandfather clock. Its length, from $T = 2\pi\sqrt{l/g}$ with $T = 2$ s and $g = 9.8$ m/s²:

$$l = \frac{gT^2}{4\pi^2} = \frac{9.8 \times 4}{4\pi^2} = \frac{9.8}{\pi^2} = 0.993 \text{ m}$$

about **99.3 cm**. Where g is smaller, it must be shorter.

What the period depends on

From $T = 2\pi\sqrt{l/g}$:

1. **Length.** A longer pendulum has a longer period: $T \propto \sqrt{l}$. Four times the length, twice the period.
2. **Acceleration due to gravity.** Where g is weaker, the period is longer: $T \propto 1/\sqrt{g}$. On the Moon, where g is a sixth of the Earth's, a pendulum swings $\sqrt{6} \approx 2.45$ times as slowly.
3. **Not the mass of the bob**, and, for small swings, **not the amplitude**. A heavier bob has a larger restoring force, but in exactly the same proportion as its larger mass, so its acceleration is the same.

Why a real pendulum is not quite simple

A point-sized bob and a weightless string can't be made, and a real pendulum falls short of the ideal in other ways too:

1. The formula holds only for **small swings**, less than about 5°. For larger angles, $\sin\theta$ is less than θ, the restoring force is smaller than the formula assumes, and the period is **longer**: about 0.4 % longer at 15°, 1.7 % at 30°.
2. The period depends on g, which changes from place to place (with latitude, height above sea level and the rocks below), so a pendulum clock keeps different time in different places.
3. The **length** must stay fixed, but heat expands the string or rod and a string can stretch; a longer pendulum runs slow. Fine clocks used rods of special alloys, or of two metals arranged so their expansions cancel.
4. **Air drag and friction** at the pivot take energy away: the amplitude dies down and, without a push from the clock's mechanism, the pendulum stops.
5. **Vibrations, wind** and other outside forces disturb it.
6. So a pendulum on its own is no good for timing over a long time; a clock must keep giving it small pushes (its escapement) to replace the energy lost.

Check 5

A pendulum clock that keeps perfect time in Kathmandu is taken to a hot place at the same height. It will:

a keep perfect time
b run fast, because heat makes it swing faster
c run slow, because its pendulum expands and swings more slowly
d stop

2.6 Angular simple harmonic motion

A body can also oscillate by **turning** back and forth about an axis. If the restoring **torque** is proportional to the **angular** displacement, the motion is **angular simple harmonic motion**, the turning form of SHM.

A **torsion pendulum** is an example: a disc or a bar hung from the middle by a wire. Turn the disc through an angle and the wire twists; the twisted wire tries to untwist, and when let go the disc swings back and forth about the wire as its axis. The springiness here is the wire's **twist**, not a spring's stretch. The balance wheel of a mechanical watch, with its hairspring, works the same way.

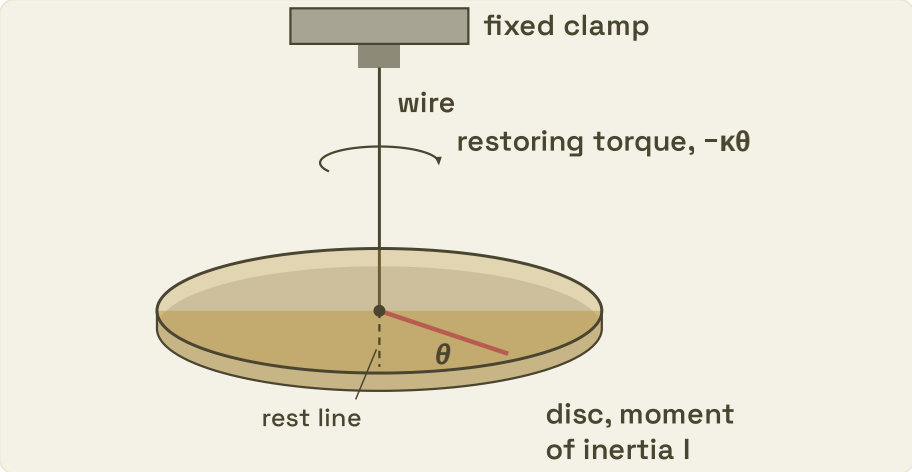


Figure 2.8 *A torsion pendulum: a horizontal disc hung at its center from a fixed support by a wire. Turned through an angle θ and let go, it turns back and forth about the wire.*

1. Turn the disc through an angle θ from its rest position. The twisted wire gives a restoring torque proportional to the twist, and opposite to it:

$$\tau = -\kappa\theta$$

where κ (kappa) is the **torsional constant** of the wire, in N m/rad.

2. With $\tau = I\alpha$ (Unit 1), I being the disc's moment of inertia about the wire:

$$I\alpha = -\kappa\theta \qquad \Rightarrow \qquad \alpha = -\frac{\kappa}{I}\theta \tag{2.8}$$

3. This has the form $a = -\omega^2 x$ of linear SHM: the angular acceleration is proportional to the angular displacement and directed towards the mean position. So the disc moves in angular SHM, with

$$\omega = \sqrt{\frac{\kappa}{I}}, \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{\kappa}} \tag{2.9}$$

A heavier disc, or one with its mass farther out, turns back and forth more slowly; a thicker, stiffer wire, faster. Unlike a pendulum, a torsion pendulum doesn't need small angles: the wire's torque stays proportional to the twist even for large turns. (Other books write the torsional constant as k or C; the meaning is the same.)

Check 6
A torsion pendulum's disc is replaced by one with four times the moment of inertia, on the same wire. The period:

a stays the same
b doubles
c is four times as long
d halves

2.7 Oscillatory motion: free, damped and forced

An **oscillatory motion** is a to-and-fro motion about a central point, like a swinging pendulum or a bouncing spring. It is described by its amplitude (how far it moves), its period (the time for one complete movement) and its frequency (how often it moves back and forth). What happens to an oscillation over time depends on what else acts on it.

Free oscillation

When a body is pulled aside and let go, and nothing else acts on it, it oscillates at a frequency set by itself alone: its **natural frequency**, f_0. A pendulum's is $\frac{1}{2\pi}\sqrt{g/l}$, a spring's $\frac{1}{2\pi}\sqrt{k/m}$. This is a **free oscillation**. An ideal free oscillation, with no friction, would keep the same amplitude forever.

Damped oscillation

Every real oscillation meets friction or air resistance, which takes energy from it. Its amplitude grows smaller and smaller until it stops. This is a **damped oscillation**, and the loss of energy is **damping**. The more damping, the faster the oscillation dies away; damping also makes each oscillation take a little longer.

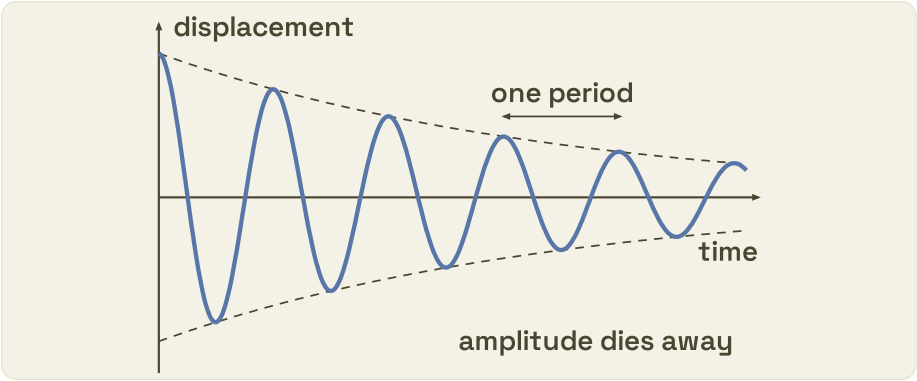


Figure 2.9 *Displacement against time for a damped oscillation. The amplitude falls by the same fraction every cycle, so the peaks lie on a curve that dies away exponentially (dashed); in air the damping is light and the oscillations last many cycles.*

There are three kinds of damping:

- **Light damping (underdamping):** the body still oscillates, but each swing is smaller than the last, by the same fraction each time. A pendulum in air and a tuning fork are lightly damped.
- **Critical damping:** just enough damping that the body returns to its mean position in the **shortest** possible time **without** overshooting. A car's shock absorbers and the needle of a good meter are designed to be close to critically damped.
- **Heavy damping (overdamping):** so much damping that the body doesn't oscillate at all; it creeps back slowly to the mean position, like a pendulum in thick syrup, or a door with a door closer.

Examples of damped oscillations:

1. **A pendulum in air:** air resistance slowly brings it to rest.
2. **Car shock absorbers:** when a car hits a bump, its springs start to oscillate; the shock absorbers (pistons in oil) damp the motion so the car settles quickly instead of bouncing for a long time.
3. **A tuning fork:** its sound dies away as the vibration loses energy to the air (that energy *is* the sound) and inside the metal.
4. **Slow-closing doors:** a door closer damps the door's swing so it closes gently instead of slamming.
5. **A spring in water:** a mass bobbing on a spring under water dies down much faster than in air.

Forced oscillation

If an outside force that varies periodically keeps acting on a body, the body is made to oscillate at the **frequency of the force** (the **driving frequency**), whatever its own natural frequency. This is a **forced oscillation** (forced vibration). After a short settling time, the body oscillates steadily, the force's push each cycle making up for the energy lost to damping. How large the oscillation is depends on the strength of the force, and very much on how close the driving frequency is to the natural frequency.

Resonance

When the driving frequency **equals the natural frequency** of a body, the body takes in energy from the force most easily and oscillates with its **largest amplitude**. This is **resonance**. Pushing a swing is the everyday example: push once each swing, at the right moment, and even small pushes build a large swing.