Unit 2

Answers: Periodic Motion

Quick checks

1 c On a spring, the restoring force is $-kx$, so the motion is SHM. A bouncing ball repeats, but between bounces only its weight acts, which doesn't grow with the distance from a mean position; the car goes round, not back and forth; the stone doesn't repeat at all.

2 b At an extreme, $y = \pm r$: $v = \omega\sqrt{r^2 - r^2} = 0$, and $|a| = \omega^2 r$, the greatest. The particle stops for an instant, while the pull back is strongest.

3 b Each is then half the total: $\frac{1}{2}m\omega^2 y^2 = \frac{1}{2} \times \frac{1}{2}m\omega^2 r^2$, so $y^2 = r^2/2$ and $y = r/\sqrt{2} \approx 0.71r$. At $y = r/2$ the potential energy is only a quarter of the total.

4 c In parallel the spring constant doubles, $2k$. $T = 2\pi\sqrt{m/k}$, so doubling k divides the period by $\sqrt{2}$.

5 c Heat lengthens the pendulum's rod; $T = 2\pi\sqrt{l/g}$, so a longer pendulum has a longer period and the clock loses time.

6 b $T = 2\pi\sqrt{I/\kappa}$: four times I gives $\sqrt{4} = 2$ times the period.

7 c Two SHMs of the same ω, a quarter of a cycle apart, add to one SHM: $3\sin\omega t + 4\cos\omega t = 5\sin(\omega t + \phi)$, with $\tan\phi = 4/3$. The amplitude is $\sqrt{3^2 + 4^2} = 5$ cm, not 3 + 4: the two never reach their greatest values at the same instant.

8 d The acceleration always points towards the mean position. The velocity points towards it on the way in (angle 0) and away from it on the way out (angle π).

9 c Starting at the extreme, $x = r\cos\omega t$. $x = r/2$ when $\cos\omega t = \frac{1}{2}$, $\omega t = \pi/3$, so $t = \dfrac{\pi/3}{2\pi/T} = \dfrac{T}{6}$. (Not T/8: the particle is slow near the extreme, so the first half of the distance takes longer than the second.)

10 c $T \propto \sqrt{l}$, so three times the length gives $\sqrt{3} \approx 1.73$ times the period.

11 a $T \propto 1/\sqrt{g}$: $T = 2 \times \sqrt{6} = 4.9$ s, about 5 s.

12 c A pendulum's period, $2\pi\sqrt{l/g}$, doesn't depend on the mass. (It stays the same as long as the second girl sits the same way, keeping the center of gravity, and so the effective length, unchanged.)

13 b $f = \dfrac{1}{2\pi}\sqrt{k/m}$. k/m becomes $\dfrac{2k}{m/2} = 4\dfrac{k}{m}$, so f becomes $\sqrt{4} = 2$ times as much.

14 c $f \propto 1/\sqrt{m}$, so doubling the mass divides the frequency by $\sqrt{2}$.

15 **b** Two equal springs in series have spring constant $k/2$ (each stretches as much as one spring alone would). $f \propto \sqrt{k}$, so the frequency becomes $f/\sqrt{2}$. (Side by side, in parallel, they would make $2k$ and $\sqrt{2}\,f$.)

16 **b** Resonance is the large response of a system driven at its own natural frequency.

17 **c** A guitar's or violin's hollow body, and the air in it, resonate with the strings and make their sound loud. The free pendulum is a free oscillation, the suspension is damping, and the stretched spring isn't oscillating at all.

18 **b** $I\alpha = -k\theta \Rightarrow \alpha = -\dfrac{k}{I}\theta = -\omega^2\theta$, so $\omega = \sqrt{k/I}$.

13.

a. **Simple harmonic motion** is a periodic motion in which the restoring force (and so the acceleration) is directly proportional to the displacement from the mean position and always directed towards it: $F = -kx$, $a = -\omega^2 x$. Its characteristics: it is periodic, to and fro about a fixed mean position; the acceleration is proportional to the displacement and opposite to it; the speed is greatest at the mean position and zero at the extremes; the acceleration is zero at the mean position and greatest at the extremes; the period does not depend on the amplitude; and with no damping the total energy stays constant.

b. Let the particle be the shadow, on a diameter, of a point going round a circle of radius r with angular velocity ω. Its displacement is $y = r\sin\omega t$. Then $v = \dfrac{dy}{dt} = r\omega\cos\omega t$, and $a = \dfrac{dv}{dt} = -r\omega^2\sin\omega t = -\omega^2 y$. So the acceleration is $-\omega^2 y$: proportional to the displacement and towards the mean position.

c. $\omega = \dfrac{2\pi}{T} = \pi$ rad/s, $r = 0.5$ m, $m = 0.1$ kg.

i. $E = \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}\times 0.1\times\pi^2\times 0.5^2 = \mathbf{0.123\ J}$.

ii. $KE = \frac{1}{2}m\omega^2(r^2 - y^2) = \frac{1}{2}\times 0.1\times\pi^2\times(0.25 - 0.09) = \mathbf{0.079\ J}$.

iii. $v = \omega\sqrt{r^2 - y^2} = \frac{1}{2}\omega r$ gives $r^2 - y^2 = \dfrac{r^2}{4}$, so $y = \dfrac{\sqrt{3}}{2}r = 0.866\times 0.5 =$ **0.433 m**.

14.

a. P to Q is half a period, so $T = 1.0$ s. From $T = 2\pi\sqrt{l/g}$, $l = \dfrac{gT^2}{4\pi^2} = \dfrac{9.8\times 1^2}{39.5} =$ **0.25 m** (24.8 cm).

b. i. Two forces act on the bob: its **weight** mg, vertically down, and the **tension** T in the string, along the string towards the point of suspension (as in Figure 2.6). Splitting the weight: $mg\cos\theta$ along the string, balanced by the tension, and $mg\sin\theta$ along the arc, towards the mean position: the restoring force.

ii. The restoring force is $-mg\sin\theta \approx -mg\dfrac{x}{l}$ for small angles, so $a = -\dfrac{g}{l}x$, SHM with $\omega = \sqrt{g/l}$. Then $f = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{l}} = \dfrac{1}{2\pi}\sqrt{\dfrac{9.8}{0.248}} = \mathbf{1.0\ Hz}$, as it must be for a period of 1 s.

15. a. The **equilibrium position** is the position where the mass hangs at rest, where the spring's upward pull equals the weight: $kx = mg$, x being the stretch. The mass oscillates about this position.

b. Displaced by y below the equilibrium position, the spring is stretched by $x + y$ and pulls up with $k(x + y)$ (Hooke's law), while the weight pulls down with $mg = kx$. The net force is $mg - k(x + y) = -ky$: proportional to the displacement from the equilibrium position and directed back towards it. Above the equilibrium position, the spring's pull is less than the weight, and the net force is downward, again $-ky$.

c. i. $k = \dfrac{mg}{x} = \dfrac{5 \times 9.8}{0.25} = \mathbf{196\ N/m}$.

ii. $T = 2\pi\sqrt{\dfrac{m}{k}} = 2\pi\sqrt{\dfrac{5}{196}} = \mathbf{1.00\ s}$.

16. a. **Simple harmonic motion** (a free oscillation of a spring–mass system; in practice lightly damped).

b. $T = 2\pi\sqrt{\dfrac{m}{k}}$.

c. At equilibrium the spring's pull balances the weight: $kh = mg$, so $\dfrac{m}{k} = \dfrac{h}{g}$. Putting this in the period: $T = 2\pi\sqrt{\dfrac{h}{g}}$, or $h = \dfrac{gT^2}{4\pi^2}$. (The period is the same as a simple pendulum's whose length equals the spring's stretch.)

17. a. **Angular SHM** is an oscillation in which a body turns back and forth about an axis, under a restoring torque proportional to its angular displacement and opposite to it ($\tau = -\kappa\theta$), so that its angular acceleration is proportional to its angular displacement and towards the mean position. Example: a torsion pendulum (a disc hung on a wire), or the balance wheel of a watch.

b. Turning the disc through θ twists the wire, which gives a restoring torque $\tau = -\kappa\theta$. With $\tau = I\alpha$: $I\alpha = -\kappa\theta$, so $\alpha = -\dfrac{\kappa}{I}\theta$. This is angular SHM with $\omega^2 = \dfrac{\kappa}{I}$, so $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{I}{\kappa}}$.

c. $T = 2\pi\sqrt{\dfrac{0.05}{0.10}} = 2\pi \times 0.707 = \mathbf{4.44\ s}$.

18. a. **Damping** is the loss of energy of an oscillating system to friction, air resistance or other resistive forces, which makes its amplitude die away. Types: **light damping** (underdamping: it oscillates, each swing smaller), **critical damping** (it returns to the mean position in the shortest time without oscillating) and **heavy damping** (overdamping: it creeps back slowly without oscillating).

b. The system resonates: it takes in energy from the force most efficiently, and its amplitude grows to its greatest value, limited only by damping. With light damping this amplitude can be very large.

c. i. Draw time across and displacement up, with the equilibrium position as the time axis. Draw a wave crossing the axis at equal intervals of time (half a period apart), its peaks on alternate sides, the first at 8.0 cm, the next on the same side, one period later, at 6.0 cm, then 4.5 cm and 3.4 cm, joined by a smooth wavy curve. The peaks lying on a falling curve (dashed through their tops) show the damping. Each peak is about 0.75 of the one before (6.0/8.0, 4.5/6.0, 3.4/4.5), so the amplitude falls by the same fraction each cycle: an exponential decay.

ii. **Slightly greater than 1.6 s.** Damping opposes the motion and slows the bob down, so each oscillation takes a little longer than without it. In air the damping is light, so the increase is very small.

iii. In oil the damping is much heavier: the peaks fall away much faster (each peak a much smaller fraction of the last), so there are far fewer oscillations before it stops; and the period becomes noticeably longer (the crossings of the axis further apart). In thick enough oil it would not oscillate at all.

19. a. **Resonance** is the large-amplitude oscillation of a system driven by a periodic force whose frequency equals the system's natural frequency. It occurs when the driving frequency equals the natural frequency ($f = f_0$, or $\omega = \omega_0$).

b. At resonance the displacement **lags** the driving force by a quarter of a cycle, $\pi/2$ (90°). The velocity is then in step with the force, so the force always pushes the way the body moves and feeds in energy at the greatest rate.

c. Sketch amplitude (up) against driving angular frequency ω (across), as in Figure 2.10: small at low ω, rising to a peak at $\omega = 5$ rad/s, and falling again at higher ω; with lighter damping the peak is taller and sharper. The amplitude is greatest at 5 rad/s because that is the oscillator's own natural frequency: each push then comes at the same point in each oscillation and adds to the motion, instead of sometimes opposing it. The resonance condition: $\omega = \omega_0 = 5$ rad/s, that is $f = \dfrac{\omega}{2\pi} = \dfrac{5}{2\pi} = \mathbf{0.80\ Hz}$, a period of 1.26 s.

20.

a. **Resonance** (the response of a forced oscillation).

b. Fb is the **natural frequency** of the system (the resonant frequency). At it, the driving force is in step with the system's own oscillation, so energy is fed in most efficiently and the amplitude is at its greatest, limited only by damping.

c. At A (Fa, below the natural frequency) and at C (Fc, above it) the force is out of step with the system's natural oscillation: for part of each cycle it pushes against the motion, so less energy is fed in and the amplitude is smaller than at B. As drawn, A and C are at about the same height, both well below B; far below the natural frequency the amplitude levels off at the force's steady stretch, and far above it, it falls towards zero.

21.

a. SHM is a periodic motion whose restoring force is proportional to the displacement from the mean position and directed towards it. For the block, at a displacement y from equilibrium (where $kx = mg$), the net force is $mg - k(x + y) = -ky$ (Hooke's law). Newton's second law gives $m\dfrac{d^2y}{dt^2} = -ky$, or $\dfrac{d^2y}{dt^2} = -\omega^2 y$ with $\omega = \sqrt{k/m}$: the equation of SHM, whose solution is $y = r\sin(\omega t + \phi)$.

b. The total mechanical energy is the sum of its kinetic and potential energies (here, measured from the equilibrium position): $PE = \frac{1}{2}ky^2$ (the work done against the restoring force ky, $\int_0^y ky\,dy$) and $KE = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(r^2 - y^2) = \frac{1}{2}k(r^2 - y^2)$. Total $E = \frac{1}{2}kr^2 = \frac{1}{2}m\omega^2 r^2$, the same at every instant. With time, the energy swaps between the two: all kinetic at the mean position, all potential at the extremes, twice each period; KE and PE each go up and down (as $\cos^2\omega t$ and $\sin^2\omega t$), while their sum stays constant.

c. Gravity gives a **constant** force, mg. The spring's stretch $x = mg/k$ at equilibrium balances it once and for all; only the extra stretch y gives an unbalanced force, $-ky$. So gravity just moves the point of oscillation down by mg/k; the restoring force about it, and the period $2\pi\sqrt{m/k}$, are the same as with no gravity.

d. i. In parallel, $k = k_1 + k_2 = 200 + 300 = \mathbf{500\ N/m}$.

ii. $T = 2\pi\sqrt{\dfrac{m}{k}} = 2\pi\sqrt{\dfrac{2}{500}} = \mathbf{0.40\ s}$ (0.397 s).

iii. $E = \frac{1}{2}kr^2 = \frac{1}{2} \times 500 \times 0.15^2 = \mathbf{5.6\ J}$ (5.625 J).

22.

i. $\omega = 2\pi f = 2\pi \times 5 = \mathbf{31.4\ rad/s}$.

ii. $y = r\sin\omega t = 0.4\sin(31.42 \times 0.1) = 0.4\sin\pi = \mathbf{0\ m}$: half a period (0.1 s, since T = 0.2 s) has passed, so it is back at the mean position.

iii. $v = r\omega\cos\omega t = 0.4 \times 31.42 \times \cos\pi = \mathbf{-12.6\ m/s}$: the greatest speed, now moving the opposite way to its start. $a = -\omega^2 y = \mathbf{0}$.

23.

i. Each cycle leaves 95 % of the amplitude: $0.5 \times 0.95^{10} = 0.5 \times 0.599 = \mathbf{0.30\ m}$.

ii. Energy is proportional to the square of the amplitude, so each cycle leaves 0.95^2 of the energy: $40 \times 0.95^{20} = 40 \times 0.358 = \mathbf{14.3\ J}$.