Unit 1

Rotational Dynamics and Oscillatory Motion

This unit online, with its interactives:
science-playground.com/s/12f1

Spin a coin on its edge on a table. It stays upright as long as it spins fast, wobbles as it slows, and falls flat once it stops. Spin a top, a fan, a potter's wheel in Bhaktapur, a bicycle wheel: each keeps turning after you let go, some longer than others, and each resists being started and stopped.

Grade 11 described how a body moves along a line and why. This unit does the same for bodies that **turn**. You will meet the turning versions of mass, force, momentum and kinetic energy, and you will find that the same laws hold, with one new twist: for turning, **where** the mass sits matters as much as how much there is. That is why a skater spins faster when she pulls her arms in, why a ring rolls down a slope more slowly than a disc of the same mass, and why the Earth has kept spinning for four and a half billion years.

1.1 Introduction

From a spinning top to the rotating Earth, turning is part of everyday life. Three questions to keep in mind as you read:

1. Why does a skater spin faster when she pulls her arms in? (Section 1.7)
2. How do wheels help vehicles move? (Rolling, after Section 1.4)
3. What keeps the Earth spinning, day after day? (Section 1.7)

Rigid bodies

A real body bends, stretches and squashes a little when it is pushed or turned. To study turning simply, we use an ideal model:

A **rigid body** is a body in which the distance between any two of its particles stays the same, whatever forces or torques act on it. Its particles keep their positions relative to one another throughout the motion.

A steel wheel, a cricket bat and a door are very nearly rigid; a rubber ball being squeezed or a jelly is not. The study of how rigid bodies move under forces and torques is **rigid body dynamics**. Engineers use it to design vehicles, cranes and machines, and the makers of computer games and animated films use it to make tumbling objects look real.

A rigid body can move in two basic ways:

- **Translation** (translatory motion): every point of the body moves along parallel paths, by the same displacement in the same time, as a box sliding across a floor or a car driving straight.
- **Rotation** (rotatory motion) about a fixed axis: every point moves round a circle centered on the axis, as a wheel, a fan or a door does.

Most motions are both at once: a rolling wheel turns about its axle while the axle moves forward.

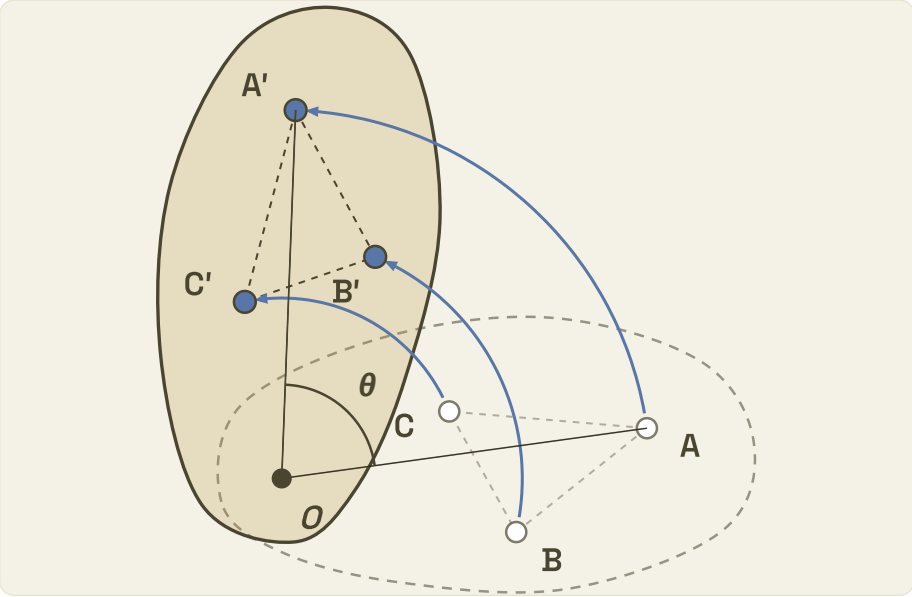


Figure 1.1 *A rigid body turning about a fixed axis. Every particle moves round a circle centered on the axis, and the distances between the particles (the dashed lines between A, B and C) never change. All the particles turn through the same angle in the same time.*

1.2 Moment of inertia

Newton's second law, $F = ma$, says that **mass** sets how hard it is to change a body's motion along a line: the larger the mass, the smaller the acceleration a given force produces. This mass is called the **inertial mass**.

Turning has a quantity that plays the same part. The **moment of inertia**, I, measures how hard it is to start a body turning about an axis, or to stop it once it turns. But unlike mass, it depends on two things:

- how much mass the body has, and
- **how far that mass is from the axis**.

The farther the mass is spread from the axis, the larger the moment of inertia and the harder the body is to spin up or slow down.

Try it with a broom. Hold it in the middle and twist it quickly back and forth about its long axis: easy. Now hold it at one end and swing it back and forth sideways: much harder, though the broom's mass hasn't changed. In the second case most of its mass is far from the axis of turning.

For the same reason, a ring is harder to spin than a solid disc of the same mass and radius: all of the ring's mass is at the rim, while much of the disc's is near the center. And a heavy car wheel, with much of its mass far out in the tyre and rim, is much harder to spin than a light bicycle wheel.

In short, **moment of inertia is the turning equivalent of mass**: a body's resistance to any change in its turning.

Defining it

Think of a body as made of many small particles of masses $m_1, m_2, \dots, m_n$ at perpendicular distances $r_1, r_2, \dots, r_n$ from the axis (Figure 1.2). The moment of inertia of the body about that axis is

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots + m_n r_n^2 = \sum_{i=1}^{n} m_i r_i^2$$

Each particle counts by its mass times the **square** of its distance from the axis. So a particle twice as far out counts four times as much. Its SI unit is the **kilogram meter squared**, **kg m²**, and it is a scalar.

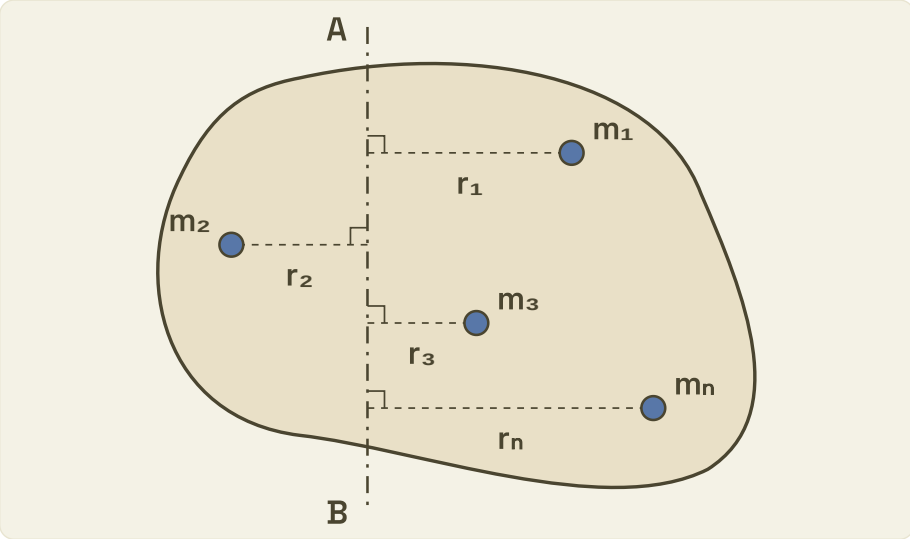


Figure 1.2 *A rigid body turning about the axis AB. Its particles m_1, m_2, m_3 … m_n are at perpendicular distances r_1, r_2, r_3 … r_n from the axis. Each adds mr^2 to the moment of inertia.*

For a single small mass M at a distance R from the axis, $I = MR^2$. A thin ring of mass M and radius R, turning about its own axis, has all its mass at the same distance R, so $I = MR^2$ too. But for a disc, a rod or a sphere, the particles are at all sorts of distances, and I is **not** simply the total mass times the square of some size of the body: we must add up mr^2 for every particle, or, for a continuous body, integrate (Section 1.3).

Check 1

Two balls of 1 kg each are fixed on a light rod, one 0.1 m and the other 0.3 m from the axis. Which ball adds more to the moment of inertia, and by how much?

a the outer one, 3 times as much
b the outer one, 9 times as much
c both add the same, since their masses are equal
d the inner one, since it is nearer the axis

The same body, a different axis

A body has a different moment of inertia about each axis, because its particles are at different distances from each one. A solid sphere of mass M and radius R has $I = \frac{2}{5}MR^2$ about a diameter, but $\frac{7}{5}MR^2$ about a tangent (a line touching its surface), three and a half times as much, because about the tangent most of its mass is much farther from the axis. So whenever you give a moment of inertia, say **about which axis**.

DID YOU KNOW?

Why a tightrope walker carries a long pole

A walker on a rope across a gorge carries a long, heavy, often drooping pole. The pole puts mass far from the rope, and so gives the walker a much larger moment of inertia about the rope. A small push or a slight lean then makes her start to topple only very slowly, giving her time to correct it. The same thing helps you balance with your arms held out. And how do you know that you are tipping? Three fluid-filled loops in each of your inner ears, the semicircular canals, sense the slightest turning of your head. When they don't work well, or you walk with your eyes closed, small turns of your body go unnoticed and it is hard to walk in a straight line.

1.2.1 Radius of gyration

It is often useful to say how far from the axis a body's mass is, "on average", in one number.

The **radius of gyration**, K, of a body about an axis is the distance from the axis at which the body's whole mass, placed at one point, would have the same moment of inertia as the body itself.

So if the body has mass M,

$$I = MK^2 \qquad \Rightarrow \qquad K = \sqrt{\frac{I}{M}}$$

Its SI unit is the meter. It depends on the axis, and on how the mass is spread about it, not on how much mass there is: double every particle's mass and I doubles too, so K stays the same.

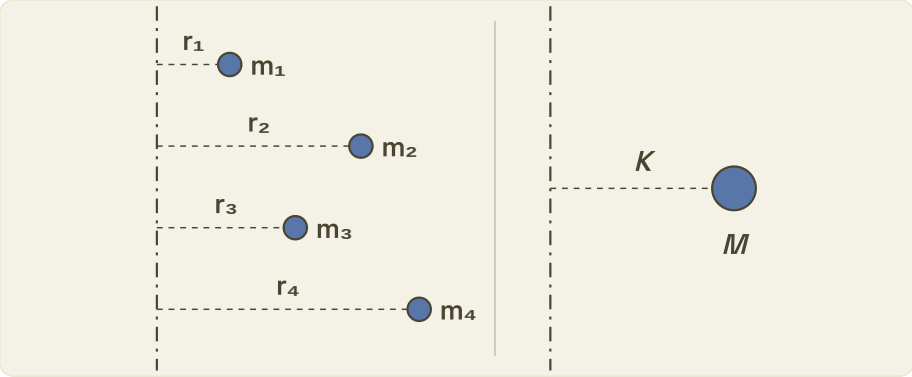


Figure 1.3 *The radius of gyration. The particles m_1, m_2, m_3 and m_4 at r_1, r_2, r_3 and r_4 from the axis have the same moment of inertia as the whole mass M placed at the single distance K.*

For n equal particles. Let the body be made of n particles, each of mass m, at distances $r_1, r_2, \ldots, r_n$ from the axis. Its total mass is $M = nm$, and

$$I = mr_1^2 + mr_2^2 + \cdots + mr_n^2 = m\left(r_1^2 + r_2^2 + \cdots + r_n^2\right)$$

Putting $I = MK^2 = nmK^2$:

$$nmK^2 = m\left(r_1^2 + r_2^2 + \cdots + r_n^2\right) \quad \Rightarrow \quad K = \sqrt{\frac{r_1^2 + r_2^2 + \cdots + r_n^2}{n}}$$

So for equal particles, the radius of gyration is the **root mean square** of their distances from the axis: square the distances, take their mean, then the square root. It is a scalar, and it gives a picture of how far out the mass is: a small K means the mass is crowded near the axis, a large K that it is spread far out. Engineers use it to judge how stiff a beam or a column is against bending and buckling.

WORKED EXAMPLE

Radius of gyration of a ring and a disc

a. A thin ring of mass 2 kg and radius 0.5 m turns about its own axis. Find its radius of gyration.

b. A solid disc of the same mass and radius has $I = \frac{1}{2}MR^2$. Find its radius of gyration.

Answer

a. All the ring's mass is at 0.5 m from the axis: $I = MR^2 = 2 \times 0.5^2 = 0.5$ kg m². Then $K = \sqrt{\frac{I}{M}} = \sqrt{\frac{0.5}{2}} =$ **0.5 m**, the ring's own radius, as it should be.

b. $I = \frac{1}{2} \times 2 \times 0.5^2 = 0.25$ kg m², so $K = \sqrt{\frac{0.25}{2}} = 0.354$ m, which is $\frac{R}{\sqrt{2}}$. The disc's mass is, "on average", nearer the axis than the ring's.

1.3 Moment of inertia of a uniform rod

For a body whose mass is spread continuously, divide it into tiny pieces, each of mass dm at a perpendicular distance r from the axis, and add up $r^2\,dm$ for all of them. The sum of infinitely many tiny pieces is an integral:

$$I = \int r^2\,dm$$

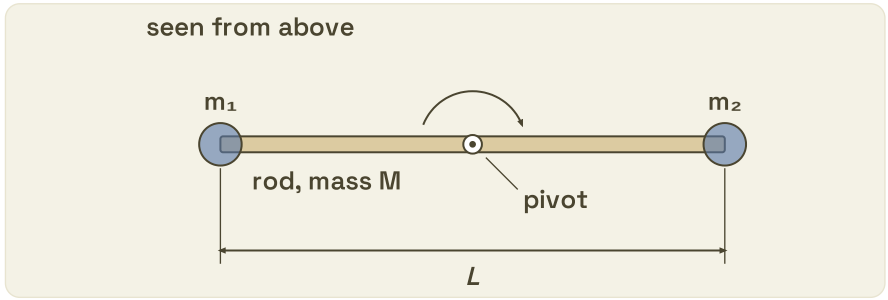


Figure 1.17 *For question 21: a uniform rod pivoted at its center with a small mass at each end.*

a. Write the expression for the total moment of inertia of the system about the central pivot, naming each term.

b. How will the moment of inertia change if one mass is moved closer to the center? Explain.

c. The rod has mass M = 3 kg and length L = 1.2 m; the point masses are $m_1 = m_2 = 1$ kg, each 0.6 m from the center. Find the total moment of inertia about the center.

22. A figure skater shows how angular momentum is conserved by changing her body position mid-spin.

a. Define angular momentum for a rotating body and give its SI unit.

b. State and explain the principle of conservation of angular momentum. Give one real-life example other than a skater.

c. A skater of moment of inertia 3 kg m² spins at 4 rad/s. She pulls in her arms, reducing her moment of inertia to 1.5 kg m². Find her new angular velocity. What does this tell us about angular momentum?

23.

a. Derive the formula for the work done by a torque in producing an angular displacement.

b. A steady torque of 8 N m is applied to a wheel. It turns through 5 rad in 2 s. Find the work done and the average power delivered.

24. A ballet dancer spins with her arms stretched out, then pulls her arms in and spins faster.

a. Name and state the physical principle that explains why her angular velocity increases.

b. Derive the relation between angular momentum and moment of inertia for a rotating body.

c. Her moment of inertia is 4 kg m² and her angular velocity 2 rad/s. She reduces her moment of inertia to 2 kg m². Find her new angular velocity.

25. A uniform rod of length L and mass M can turn in a horizontal plane about an axis through its center, perpendicular to its length. Two small masses m_1 and m_2 are fixed at distances d_1 and d_2 from the center.

a. What is the significance of moment of inertia in rotational motion?

b. Write the formula for the moment of inertia of the uniform rod about this axis.

c. State the principle of conservation of angular momentum with a real-life example.

d. Find the total moment of inertia of the system (rod and masses) about the axis. If the system starts from rest and turns under a steady torque τ, find i. its angular acceleration α, ii. its angular velocity after turning through an angle θ.

Given: M = 6 kg, L = 2 m, $m_1 = 2$ kg at $d_1 = 0.5$ m, $m_2 = 3$ kg at $d_2 = 0.8$ m, $\tau = 12$ N m, $\theta = \pi$ rad. (Take $I_{\text{rod}} = \frac{1}{12}ML^2$.)

26. A figure skater of moment of inertia I_1 spins at angular velocity ω_1 with her arms stretched out. She then pulls her arms in, reducing her moment of inertia to I_2.

a. Define angular momentum and state its SI unit.

b. State the principle of conservation of angular momentum.

c. Explain why the skater spins faster after pulling her arms in.

d. Given $I_1 = 5$ kg m², $\omega_1 = 2$ rad/s and $I_2 = 2$ kg m², find i. her new angular velocity ω_2, ii. the change in her rotational kinetic energy ($KE = \frac{1}{2}I\omega^2$).

A companion to your textbook, not a replacement for it: study from the book your school uses, and where the two differ, follow your teacher. Found a mistake? Please tell us.

17. The torque needed to turn a rigid body plays a central part in its dynamics.
    i. Define torque and give its SI unit. Write its vector formula and explain each term.
    ii. State the relation between torque and angular acceleration. What is the rotational analog of Newton's second law?
    iii. A steady torque of 1000 N m turns a wheel of moment of inertia 200 kg m² about an axis through its center. Find its angular velocity after 3 s, starting from rest.

18. A body turns with a constant angular acceleration under a constant torque.
    a. Define angular acceleration. Write the equations of rotational motion for constant angular acceleration.
    b. Distinguish between angular displacement, angular velocity and angular acceleration.
    c. A disc starts from rest and speeds up uniformly to 20 rad/s in 5 s. Find i. its angular acceleration, ii. the angle it turns through in this time.

19. Suppose a flywheel turns about its center, with masses attached symmetrically at different distances.
    a. Define center of mass. How does its position affect the stability of a turning body?
    b. Explain rotational kinetic energy. How is it different from translational kinetic energy?
    c. Two masses, 1 kg and 2 kg, are fixed on a light (massless) rod, at 0.2 m and 0.1 m from its center O, as in the figure below. Find the moment of inertia of the system about an axis through O, perpendicular to the rod.

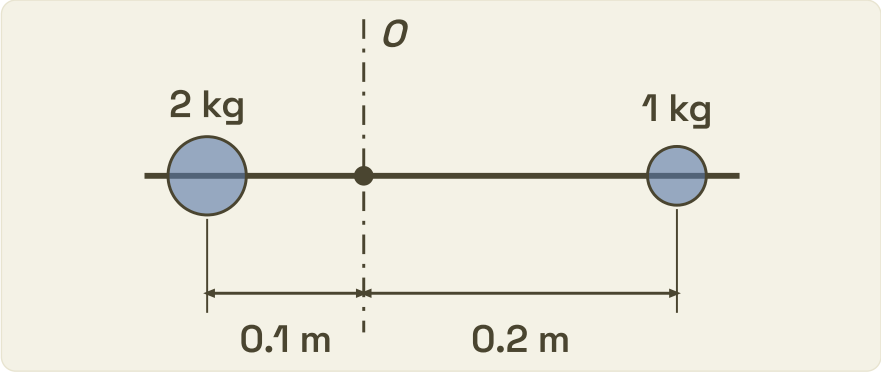


Figure 1.16 For question 19c.

20. Long rods turning about different axes behave differently, depending on how their mass is spread.
    a. Why is the moment of inertia called the rotational analog of mass?
    b. Derive the expression for the moment of inertia of a uniform rod about an axis through its center, perpendicular to its length.
    c. A rod of mass 5 kg and length 1 m turns about its center. Find its moment of inertia, and its rotational kinetic energy when it turns at 12 rad/s.

21. A uniform rod of length L and mass M is pivoted at its center and can turn freely in a horizontal plane. Two small masses, m_1 and m_2, are fixed at equal distances from the center, at its two ends.

1.3.1 About an axis through its center, perpendicular to its length

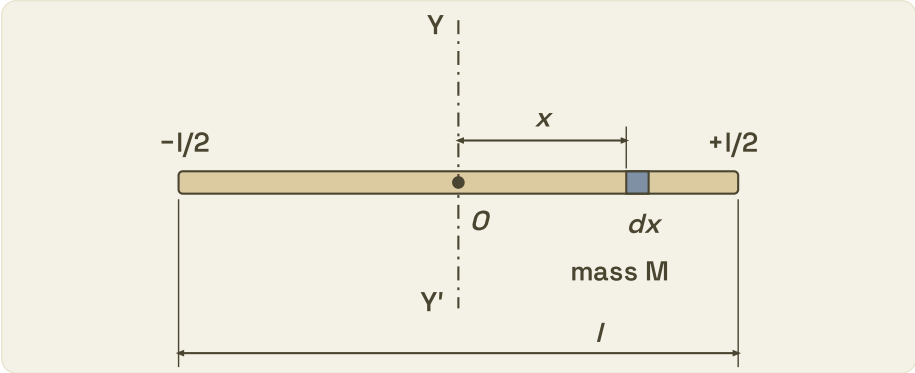


Figure 1.5 A thin uniform rod of mass M and length l, turning about an axis YY′ through its center O, perpendicular to its length. A tiny piece of length dx, at a distance x from O, has mass dm = (M/l) dx.

Take a thin uniform rod of mass M and length l. Put the origin O at its center, with the rod along the x-axis, so the rod runs from $x = -\frac{l}{2}$ to $x = +\frac{l}{2}$.

1. The rod is uniform, so its **mass per unit length** is $\frac{M}{l}$.
2. A tiny piece of length dx at a distance x from O therefore has mass $dm = \frac{M}{l}\,dx$.
3. Its moment of inertia about the axis is

$$dI = dm\,x^2 = \frac{M}{l}x^2\,dx \tag{1.1}$$

4. The rod lies on both sides of O, so we add up (integrate) from $-\frac{l}{2}$ to $+\frac{l}{2}$:

$$I = \int_{-l/2}^{l/2} \frac{M}{l}x^2\,dx = \frac{M}{l}\left[\frac{x^3}{3}\right]_{-l/2}^{l/2} = \frac{M}{l}\left(\frac{l^3}{24} + \frac{l^3}{24}\right)$$

$$\boxed{I = \frac{Ml^2}{12}} \tag{1.2}$$

5. If K is its radius of gyration, $MK^2 = \frac{Ml^2}{12}$, so

$$K = \frac{l}{\sqrt{12}} = \frac{l}{2\sqrt{3}} \approx 0.29\,l \tag{1.3}$$

1.3.2 About an axis through one end, perpendicular to its length

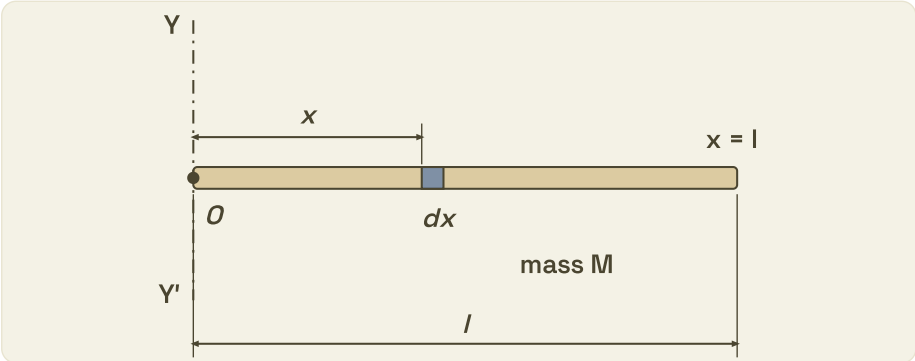


Figure 1.6 The same rod turning about an axis YY′ through its end O. Now every piece is on one side of the axis, from x = 0 to x = l.

Now put the origin at one end of the rod, so the rod runs from $x = 0$ to $x = l$.

1. As before, a piece of length dx at a distance x from O has mass $dm = \dfrac{M}{l}\,dx$, and

$$dI = dm\,x^2 = \frac{M}{l}x^2\,dx \tag{1.4}$$

2. All the mass is on one side of the axis, so we integrate from 0 to l:

$$I = \int_0^l \frac{M}{l}x^2\,dx = \frac{M}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{M}{l}\times\frac{l^3}{3}$$

$$\boxed{I = \frac{Ml^2}{3}} \tag{1.5}$$

3. Its radius of gyration: $MK^2 = \dfrac{Ml^2}{3}$, so

$$K = \frac{l}{\sqrt{3}} \approx 0.58\,l \tag{1.6}$$

About its end, the rod's moment of inertia is **four times** as large as about its center: the same rod, the same mass, but now half of it is much farther from the axis. That is why a cricket bat or a hockey stick is so much harder to swing when held at the very end than when held near its middle.

CAREFUL

Check with the parallel axis theorem

A rule from higher physics links the two results: the moment of inertia about any axis equals the moment of inertia about a parallel axis through the center of mass, plus the mass times the square of the distance d between the axes: $I = I_{\text{cm}} + Md^2$. For the rod, the end is $d = \frac{l}{2}$ from the center, so $I_{\text{end}} = \dfrac{Ml^2}{12} + M\left(\dfrac{l}{2}\right)^2 = \dfrac{Ml^2}{12} + \dfrac{3Ml^2}{12} = \dfrac{Ml^2}{3}$, as we found. The sphere's tangent result above comes the same way: $\frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2$.

Moments of inertia of some common bodies

The same kind of integration gives the moments of inertia of other bodies. You will need these results in this unit (you need not derive the ones for discs and spheres yet):

Body (mass M)	Axis	Moment of inertia	K^2/R^2
Thin ring or hollow cylinder, radius R	its own axis	MR^2	1
Hollow sphere (thin shell), radius R	a diameter	$\frac{2}{3}MR^2$	2/3
Solid disc or solid cylinder, radius R	its own axis	$\frac{1}{2}MR^2$	1/2
Solid sphere, radius R	a diameter	$\frac{2}{5}MR^2$	2/5
Thin rod, length l	through its center, perpendicular	$\frac{1}{12}Ml^2$	
Thin rod, length l	through one end, perpendicular	$\frac{1}{3}Ml^2$	

For the same mass and radius, the order from largest to smallest is: **ring > hollow sphere > disc > solid sphere**, the same order as how much of each one's mass lies far from the axis.

Check 2

A ring, a solid disc and a solid sphere have the same mass and radius. Which has the largest moment of inertia about its axis of symmetry?

a the ring
b the disc
c the solid sphere
d all three are equal

1.4 Rotational kinetic energy of a rigid body

A spinning wheel that stays in one place still has kinetic energy: each of its particles is moving. Let us add their kinetic energies.

Check 15

10. A ring, a solid sphere, a disc and a hollow sphere all have the same mass and radius. Which has the greatest moment of inertia?

a the ring
b the solid sphere
c the disc
d the hollow sphere

Check 16

11. Why does a spinning gyroscope stay stable and resist changes in the direction of its axis?

a because of its moment of inertia alone
b because of its angular velocity alone
c because of the conservation of angular momentum
d because of its mass distribution alone

Check 17

12. The mass of a rotating object is drawn in so that its radius is halved, its mass unchanged and no outside torque acting. What happens to its angular momentum?

a it remains the same
b it is doubled
c it is halved
d it is quadrupled

Check 18

13. What is the angular velocity of the minute hand of a clock, in rad/s?

a π/60
b π/180
c π/1800
d π/360

Answer these questions

14. While watching a gymnast spin, you might wonder how she controls how fast she turns.
    a. Define moment of inertia and explain how it depends on the distribution of mass.
    b. State the principle of conservation of angular momentum. Explain why a gymnast spins faster when she pulls in her arms.
    c. In a stage show, a dancer of mass 60 kg spins on a frictionless turntable with her arms stretched out. Her moment of inertia is 5 kg m² and her angular velocity 2 rad/s. She then pulls her arms in, reducing her moment of inertia to 2 kg m² in 0.5 s. Find:
    i. her new angular velocity;
    ii. the change in her rotational kinetic energy;
    iii. the average torque an outside agent would need to give her the same speed-up, from 2 rad/s to her new angular velocity in 0.5 s, if she kept her arms out (I = 5 kg m²).

15. The motion of a wheel rolling down an incline without slipping combines rotation and translation.
    a. Define rolling motion and state the condition for pure rolling.
    b. Derive the relation between the linear velocity and the angular velocity of a body in pure rolling.
    c. A uniform rod of length 1.2 m and mass 2.4 kg lies on a smooth horizontal surface. A force of 12 N is applied at one end of the rod, perpendicular to it. Find:
    i. the torque about the rod's center of mass;
    ii. the angular acceleration of the rod about its center of mass;
    iii. the linear acceleration of its center of mass.

16. A rigid body turns about a fixed axis, and the way its mass is spread affects its rotational inertia.
    a. Define radius of gyration. How is it related to the moment of inertia?
    b. Explain how the moment of inertia differs for bodies such as a ring, a disc and a sphere (any two, with reasons).
    c. A uniform rod of mass 2 kg and length 1 m lies on a smooth inclined plane at 30° to the horizontal. It is hinged at one end to the plane, with the hinge's axis at right angles to the plane, so the rod can swing round in the plane of the incline. It is held lying across the slope (horizontal), then released, and starts to swing down the slope. Find, just after it is released:
    i. the torque on the rod about the hinge;
    ii. its angular acceleration. (g = 9.8 m/s²)

Exercise

Choose the correct option

Check 6

1. A disc and a ring, both with the same mass and radius, are released together from the top of an inclined plane and roll down it. Which reaches the bottom first?

a the disc
b the ring
c both reach together
d it depends on the angle of the incline

Check 7

2. Two bodies of masses m_1 and m_2 move in circles of radii r_1 and r_2. If they go round their circles in equal times, the ratio of their linear speeds $v_1 : v_2$ is:

a m_1/m_2
b r_1/r_2
c 1
d $m_1 r_1/(m_2 r_2)$

Check 8

3. A figure skater pulls in her arms during a spin. What happens to her moment of inertia and angular velocity?

a moment of inertia decreases, angular velocity increases
b moment of inertia decreases, angular velocity decreases
c moment of inertia increases, angular velocity increases
d moment of inertia increases, angular velocity decreases

Check 9

4. A ring, a solid sphere, a disc and a solid cylinder all have the same mass and radius. Which has the greatest moment of inertia about its axis of symmetry?

a the ring
b the solid sphere
c the disc
d the solid cylinder

Check 10

5. Which of these does not affect the moment of inertia of a rotating body?

a the mass of the body
b the distribution of mass about the axis
c the shape of the body
d the temperature of the body

Check 11

6. Why does a diver tuck in her body during a dive to increase her rate of turning?

a to increase her moment of inertia
b to decrease her moment of inertia
c to increase the torque on her
d to decrease the torque on her

Check 12

7. Two wheels of the same mass and shape but different radii roll without slipping on a level road at the same speed. How do their rotational kinetic energies compare?

a the larger wheel has more
b the smaller wheel has more
c they are the same
d the one with the greater angular speed has more

Check 13

8. What are the parts of the total kinetic energy of a solid cylinder rolling down an incline without slipping?

a only translational
b only rotational
c half translational, half rotational
d two thirds translational, one third rotational

Check 14

9. What can you say about the torque when the angle between $\vec{r}$ and $\vec{F}$ is 90°?

a it is zero
b it is maximum
c it is minimum
d it is constant

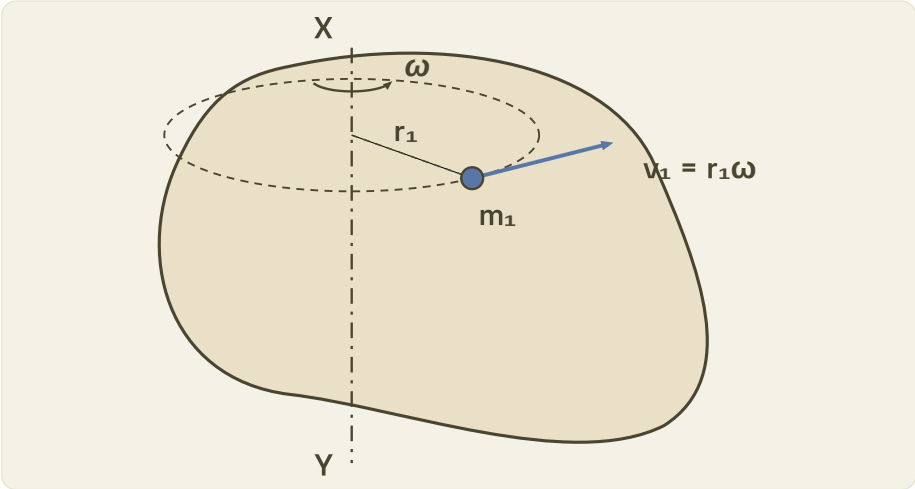


Figure 1.7 *A rigid body turning with angular velocity ω about the axis XY. The particle m_i at r_i from the axis moves along its circle with speed $v_i = r_i\omega$; a particle farther out moves faster.*

Let a rigid body turn with angular velocity ω about an axis XY. Its particles of masses $m_1, m_2, \ldots, m_n$ are at perpendicular distances $r_1, r_2, \ldots, r_n$ from the axis and move with speeds $v_1, v_2, \ldots, v_n$ round their circles.

1. The body is rigid, so every particle turns through the same angle in the same time: all have the **same** angular velocity ω.
2. A particle at distance r_i moves round its circle with speed $v_i = r_i\omega$ (Grade 11, Unit 6), so its kinetic energy is
$$KE_i = \tfrac{1}{2}m_i v_i^2 = \tfrac{1}{2}m_i(r_i\omega)^2 = \tfrac{1}{2}m_i r_i^2\omega^2$$
3. The body's rotational kinetic energy is the sum over all its particles; ω is the same for all, so it comes outside the sum:
$$KE_{\text{rot}} = \sum_{i=1}^{n} \tfrac{1}{2}m_i r_i^2\omega^2 = \tfrac{1}{2}\omega^2 \sum_{i=1}^{n} m_i r_i^2$$
4. The sum $\sum m_i r_i^2$ is the moment of inertia I of the body about XY. So
$$\boxed{KE_{\text{rot}} = \tfrac{1}{2}I\omega^2} \tag{1.7}$$

Compare it with $\frac{1}{2}mv^2$: the moment of inertia takes the place of the mass, and the angular velocity the place of the velocity. The greater the moment of inertia, the more kinetic energy a body has at a given angular velocity, and the more work it takes to spin it up from rest.

The formula also gives a meaning to I: if $\omega = 1$ rad/s, then $KE = \frac{1}{2}I$, so **the moment of inertia is numerically twice the rotational kinetic energy of the body when it turns at 1 rad/s**.

WORKED EXAMPLE

Example 1.1: a rod turning about its end

A uniform rod of length 1.2 m and mass 3 kg turns about an axis through one end, perpendicular to its length.

a. Find its moment of inertia about this axis.
b. If it turns with an angular velocity of 5 rad/s, find its kinetic energy.

Answer

Length $l = 1.2$ m, mass $M = 3$ kg.
a. For an axis through one end, perpendicular to the rod,
$I = \dfrac{Ml^2}{3} = \dfrac{3 \times 1.2^2}{3} = \mathbf{1.44\ kg\ m^2}$.
b. $\omega = 5$ rad/s:
$KE = \frac{1}{2}I\omega^2 = \frac{1}{2} \times 1.44 \times 5^2 = \mathbf{18\ J}$.

YOUR TURN

A spinning disc

A grinding wheel is a solid disc of mass 4 kg and radius 0.1 m. It turns at 3000 revolutions per minute. Find its kinetic energy.

Answer

$I = \frac{1}{2}MR^2 = \frac{1}{2} \times 4 \times 0.1^2 = 0.02$ kg m².

$\omega = 2\pi \times \dfrac{3000}{60} = 100\pi = 314$ rad/s.

$KE = \frac{1}{2} \times 0.02 \times 314^2 =$ **986 J**, about 1 kJ: as much as a 50 kg student running at 6.3 m/s. This is why a grinding wheel that bursts is so dangerous.

DID YOU KNOW?

Storing energy in a spinning wheel

A heavy wheel, a **flywheel**, stores energy as $\frac{1}{2}I\omega^2$. Old factory engines drove a huge flywheel that kept turning smoothly between the piston's strokes; a potter's kick wheel does the same between kicks. Modern flywheels of carbon fiber spin at tens of thousands of revolutions per minute in a vacuum, storing energy to steady electricity grids, and some racing cars store their braking energy in one to use again when they speed up.



Figure 1.8 *A flywheel from an old factory engine, now on display. Its heavy rim, far from the axle, gives it a large moment of inertia, so it stores a lot of energy and kept the engine turning steadily between strokes.*

Photo: Wikimedia Commons, Public domain

Rolling

A wheel rolling along a road both turns and moves forward. This section adds what you need about rolling, which the exercise asks about.

Rolling is the motion of a round body (a wheel, a ball, a cylinder) that turns about its axis while its axis moves forward along a surface. It is **pure rolling** (rolling without slipping) when the point of the body touching the surface is, at that instant, **at rest** relative to the surface: the body neither skids nor spins on the spot.

The relation between v and ω. Let a wheel of radius R roll without slipping, turning at angular velocity ω.

1. In one full turn, every point of the rim touches the ground once, so the wheel lays its whole rim, $2\pi R$, along the ground: its center moves forward $2\pi R$.
2. One turn takes the period $T = \dfrac{2\pi}{\omega}$.
3. So the center's speed is

$$v = \frac{2\pi R}{T} = \frac{2\pi R}{2\pi/\omega} \qquad \Rightarrow \qquad \boxed{v = R\omega}$$

Seen another way: a rim point moves forward with the center at v and, at the bottom, backward round the center at $R\omega$. It is at rest only if $v - R\omega = 0$, the same condition. So **pure rolling means** $v = R\omega$; if $v > R\omega$ the wheel skids, if $v < R\omega$ it spins and slips, as a bus's wheels do on mud.

Kinetic energy of rolling. A rolling body has both kinds of kinetic energy, of its center's motion and of its turning:

$$KE = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}MK^2\frac{v^2}{R^2} = \tfrac{1}{2}Mv^2\left(1 + \frac{K^2}{R^2}\right)$$



Figure 1.15 *A toy gyroscope. While its heavy wheel spins fast, it balances on a point or on a string, and resists any change in the direction of its axis.*

Photo: Eeleyeg, CC0

IN SHORT

Key ideas of Unit 1

- A **rigid body** keeps the distances between its particles fixed. Its motion can be translation, rotation, or both.
- **Moment of inertia** $I = \sum mr^2 = \int r^2\,dm$ (kg m²): the turning equivalent of mass, depending on the mass and its distance from the axis.
- **Radius of gyration** $K = \sqrt{I/M}$; for n equal particles, the root mean square of their distances.
- Uniform rod: $I = \dfrac{Ml^2}{12}$ about its center, $\dfrac{Ml^2}{3}$ about one end. Ring MR^2, disc $\frac{1}{2}MR^2$, solid sphere $\frac{2}{5}MR^2$.
- $KE_{\text{rot}} = \frac{1}{2}I\omega^2$. Rolling without slipping: $v = R\omega$, $KE = \frac{1}{2}Mv^2\left(1 + K^2/R^2\right)$.
- Torque $\vec{\tau} = \vec{r} \times \vec{F}$, $\tau = rF\sin\theta$ (N m); $\tau = I\alpha$.
- $W = \tau\theta$, $P = \tau\omega$; $\omega = \omega_0 + \alpha t$, $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$, $\omega^2 = \omega_0^2 + 2\alpha\theta$.
- Angular momentum $\vec{L} = \vec{r} \times \vec{p}$; $L = I\omega$ (kg m²/s); $\tau = \dfrac{dL}{dt}$. With no external torque, $I_1\omega_1 = I_2\omega_2$.



Figure 1.14 *A skater in a fast upright spin, arms drawn in: a small moment of inertia and a large angular velocity. His hair, free to move, is flung outward.*
Photo: Caroline Paré, CC BY-SA 3.0

Check 5
A skater spinning at 2 turns a second pulls her arms in, halving her moment of inertia. If no outside torque acts, her new rate of spin and her kinetic energy are:
a 1 turn a second, KE unchanged
b 4 turns a second, KE unchanged
c 4 turns a second, KE doubled
d 2 turns a second, KE halved

A STORY FROM SCIENCE
Who named the moment of inertia

Newton's laws are the foundation of everything in this unit, but Newton himself worked almost only with particles. The turning of a rigid body was worked out in the 1700s, mostly by the Swiss mathematician **Leonhard Euler** (1707–1783), who in 1765 gave the name *moment of inertia* to the sum $\sum mr^2$ and wrote the equations for a spinning, tumbling body. Euler went blind in his last years and kept working, dictating his mathematics to his sons and assistants; about half of all he wrote came after he lost his sight.

Linear and rotational motion compared

Every quantity of straight-line motion has a turning partner, and every law has a turning form:

Linear motion	Rotational motion
Displacement, s	Angular displacement, θ
Velocity, v	Angular velocity, ω ($v = r\omega$)
Acceleration, a	Angular acceleration, α ($a = r\alpha$)
Inertial mass, m	Moment of inertia, $I = \sum mr^2$
Linear momentum, $p = mv$	Angular momentum, $L = I\omega$
Force, $F = ma = \dfrac{dp}{dt}$	Torque, $\tau = I\alpha = \dfrac{dL}{dt}$
Work, $W = Fs$	Work, $W = \tau\theta$
Kinetic energy, $\frac{1}{2}mv^2$	Rotational kinetic energy, $\frac{1}{2}I\omega^2$
Power, $P = Fv$	Power, $P = \tau\omega$
$v = u + at$	$\omega = \omega_0 + \alpha t$
$s = ut + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$
Conservation of momentum (no external force)	Conservation of angular momentum (no external torque)

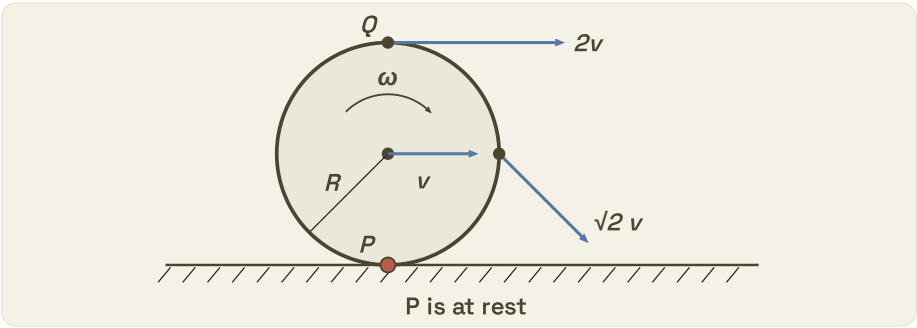


Figure 1.9 *A wheel rolling without slipping. Its center moves forward at v = Rω. Each point of the rim also turns round the center at speed Rω. At the bottom, P, the two cancel, so the point touching the ground is at rest; at the top, Q, they add to 2v.*

For a solid cylinder, $\dfrac{K^2}{R^2} = \dfrac{1}{2}$: its translational energy is $\frac{1}{2}Mv^2$ and its rotational energy $\frac{1}{4}Mv^2$, so of its total energy, **two thirds is translational and one third rotational**.

Rolling down a slope. A body rolls from rest down a slope, through a height h. The energy it gains, Mgh, is shared between moving and turning:

$$Mgh = \tfrac{1}{2}Mv^2\left(1 + \frac{K^2}{R^2}\right) \quad \Rightarrow \quad v = \sqrt{\frac{2gh}{1 + K^2/R^2}}$$

The bigger $\dfrac{K^2}{R^2}$, the more of the energy goes into turning, and the slower the body moves at the bottom. Its acceleration down a slope at θ is $a = \dfrac{g\sin\theta}{1 + K^2/R^2}$. Neither result has M or R in it: only the **shape**, through K^2/R^2, decides the race.

Body	K^2/R^2	Speed after falling h	Order
Solid sphere	2/5	$\sqrt{10gh/7} = 1.20\sqrt{gh}$	first
Solid disc or cylinder	1/2	$\sqrt{4gh/3} = 1.15\sqrt{gh}$	second
Hollow sphere	2/3	$\sqrt{6gh/5} = 1.10\sqrt{gh}$	third
Ring or hollow cylinder	1	$\sqrt{gh}$	last

A body sliding without friction would reach $\sqrt{2gh} = 1.41\sqrt{gh}$, faster than any of them, because none of its energy goes into turning.

Check 3
A solid disc and a ring of the same mass and radius roll from rest down the same slope, starting together. Which reaches the bottom first?
a the disc
b the ring
c they arrive together
d it depends on the angle of the slope

1.5 Torque and angular acceleration of a rigid body

Torque

To make a door turn, you push it far from its hinges, at right angles to it. Push near the hinge, or along the door towards the hinge, and nothing turns. What makes a body turn is not the force alone but its **torque** (moment of force), which depends on the force, where it acts, and its direction.

The **torque** of a force about an axis is the product of the force and the perpendicular distance from the axis to the line of action of the force. If a force F acts at a point whose position from the axis is r, at an angle θ to r,

$$\tau = rF\sin\theta$$

As a vector it is the cross product of the position vector and the force:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

where $\vec{r}$ is the position vector of the point where the force acts, measured from the axis (or point) about which we take the torque, and $\vec{F}$ is the force. Its size is $rF\sin\theta$; its direction is along the axis, perpendicular to both $\vec{r}$ and $\vec{F}$, given by the right-hand rule (curl the fingers of your right hand the way the force turns the body; your thumb points along $\vec{\tau}$). The SI unit of torque is the **newton meter, N m**.

- When $\theta = 90^\circ$ (force at right angles to $\vec{r}$), $\sin\theta = 1$ and the torque is **greatest**, $\tau = rF$.
- When $\theta = 0^\circ$ or 180° (force along $\vec{r}$, towards or away from the axis), the torque is **zero**.

Check 4

A spanner 0.25 m long turns a nut. A force of 40 N is applied at its end at 30° to the spanner. The torque on the nut is:

a 10 N m
b 5 N m
c 8.7 N m
d 160 N m

Torque and angular acceleration: τ = Iα

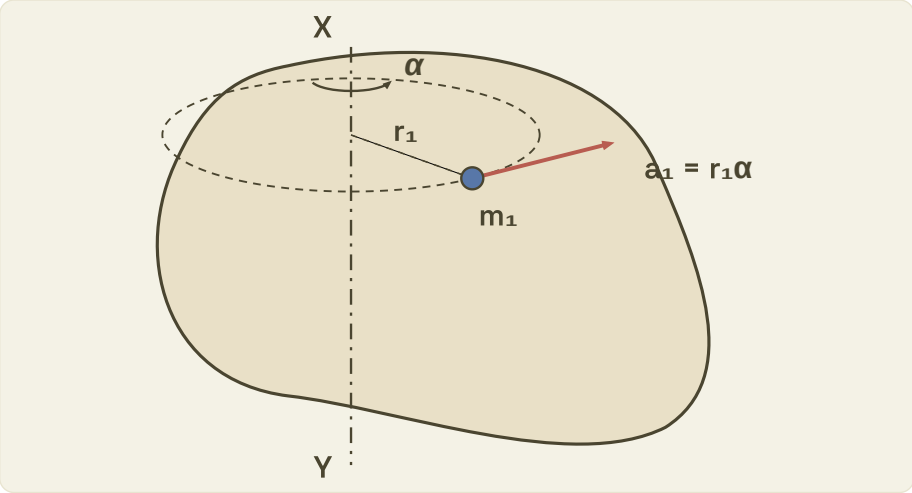


Figure 1.10 *A rigid body turning about the axis XY with angular acceleration α. Every particle has the same α, but a particle farther out has a larger linear acceleration, $a = r\alpha$.*

Let a torque act on a rigid body that can turn about a fixed axis XY, giving it a uniform angular acceleration α. Think of the body as n particles of masses $m_1, m_2, \ldots, m_n$ at perpendicular distances $r_1, r_2, \ldots, r_n$ from the axis.

1. The body is rigid, so every particle has the **same** angular acceleration α. But their linear (tangential) accelerations differ, since a particle farther out moves round a bigger circle:

$$a_1 = r_1\alpha, \quad a_2 = r_2\alpha, \quad \ldots, \quad a_n = r_n\alpha$$

2. By Newton's second law, the tangential force on the ith particle is

$$F_i = m_i a_i = m_i r_i \alpha$$

3. This force acts along the tangent, at right angles to r_i, so its torque about the axis is

$$\tau_i = F_i r_i = m_i r_i^2 \alpha \tag{1.8}$$

4. The total torque on the body is the sum over all its particles (the internal forces between particles cancel in pairs, so only the outside torque is left):

$$\tau = \sum_{i=1}^{n} m_i r_i^2 \alpha = \left(\sum_{i=1}^{n} m_i r_i^2\right)\alpha \tag{1.9}$$

5. The bracket is the moment of inertia I, so

$$\boxed{\tau = I\alpha} \tag{1.10}$$

This is the **turning form of Newton's second law**: torque does for turning what force does for straight-line motion, and moment of inertia plays the part of mass. A given torque gives a body with a large moment of inertia only a small angular

For a rigid body turning about a fixed axis, $L = I\omega$ and $\tau = \dfrac{d(I\omega)}{dt} = I\dfrac{d\omega}{dt} = I\alpha$, which is Section 1.5's law again. A torque can change how fast a body spins, or the direction of its axis, or both.

The principle of conservation of angular momentum

If no outside torque acts on a body or a system, its total angular momentum stays constant:

$$\tau = 0 \quad \Rightarrow \quad \frac{dL}{dt} = 0 \quad \Rightarrow \quad L = I\omega = \text{constant}$$

So if the body changes its moment of inertia from I_1 to I_2, by moving its mass in or out, its angular velocity changes from ω_1 to ω_2 with

$$\boxed{I_1\omega_1 = I_2\omega_2}$$

This is the **principle of conservation of angular momentum**: in the absence of an external torque, the total angular momentum of a system remains constant. A smaller moment of inertia means a faster spin, and a larger one a slower spin.

You can try this on a turning chair. Sit with a heavy book or a bottle of water in each hand, arms held out, and have a friend give you a gentle spin, then let go. Pull your hands in to your chest: you spin faster. Stretch them out again: you slow down. In an ideal chair with no friction, you could go on speeding up and slowing down like this for as long as you liked, your angular momentum always the same.

Examples of the conservation of angular momentum

1. **A skater's spin.** A spinning ice skater pulls her arms (and a leg) in close to her body: her moment of inertia falls, so her angular velocity rises, and she spins faster, up to about five turns a second in a fast spin. To slow down she stretches her arms out again. Throughout, $I_1\omega_1 = I_2\omega_2$.
2. **A planet round the Sun.** The Sun's pull on a planet points straight at the Sun, so it gives no torque about the Sun, and the planet's angular momentum about the Sun, mvr, stays constant. When the planet is nearer the Sun (smaller r), it moves faster along its orbit; when it is farther away, it moves more slowly. The Earth moves at about 30.3 km/s in early January, when it is nearest the Sun, and 29.3 km/s in early July. This is Kepler's second law.
3. **A tornado or a whirlpool.** Air spiraling in towards the center of a tornado moves to a smaller radius and so goes round faster, like the skater's arms. Water swirling into a drain speeds up the same way.
4. **A bicycle wheel.** A light wheel, with thin spokes holding a light rim, has a small moment of inertia for its size, so the rider's pedaling spins it up quickly. Since most of a wheel's mass is in its rim and tyre, far from the axle, racing cyclists value light rims most of all. The spokes are what make such a light wheel possible: thin wires in tension hold the rim true and strong without the mass of a solid disc.
5. **Melting ice and the length of the day.** When ice near the poles melts, the water spreads over the oceans, much of it nearer the equator and so farther from the Earth's axis. The Earth's moment of inertia grows, so its spin slows very slightly and the day gets longer. Since 2000, measurements show, melting ice has been lengthening the day by about 1.3 milliseconds per century, and by 2100 it could be about twice that (Kiani Shahvandi and others, 2024).
6. **A diver.** A diver leaves the board with some angular momentum. Pulling the arms and legs in (a tuck) reduces her moment of inertia and makes her somersault faster; stretching out before entering the water slows the turning so she enters cleanly.
7. **A gyroscope.** A fast-spinning wheel keeps the direction of its axis, because changing it would need a torque to change its angular momentum. That is why a spinning top stands up, a moving bicycle is easier to balance than a still one (partly), and gyroscopes keep spacecraft, drones and ship compasses pointing steadily.

YOUR TURN

Another propeller

An aircraft engine delivers 300 hp to its propeller, which turns at a steady 1800 rpm. Find a. the torque on the propeller, b. the work done in one revolution. (1 hp = 745.7 W)

Answer

$P = 300 \times 745.7 = 2.237 \times 10^5$ W; $f = 30$ rev/s, $\omega = 60\pi = 188.5$ rad/s.

a. $\tau = \dfrac{P}{\omega} = \dfrac{2.237 \times 10^5}{188.5} = \mathbf{1187\ N\,m}$.

b. $W = \tau \times 2\pi = \dfrac{P}{f} = \dfrac{2.237 \times 10^5}{30} = \mathbf{7457\ J}$, about 7460 J.

1.7 Angular momentum and its conservation

Angular momentum

A moving body has linear momentum, $\vec{p} = m\vec{v}$. A turning body has its turning equivalent.

The **angular momentum** of a particle about a point is the moment of its linear momentum about that point:

$$\vec{L} = \vec{r} \times \vec{p}$$

where $\vec{r}$ is the particle's position vector from the point and $\vec{p} = m\vec{v}$ its momentum. Its size is $L = rp\sin\theta$, and for a particle going round a circle of radius r (where $\vec{v}$ is at right angles to $\vec{r}$), $L = mvr$. Its SI unit is the **kilogram meter squared per second, kg m²/s** (the same as J s).

For a rigid body turning about a fixed axis, add up the particles. A particle at r_i moves round its circle at $v_i = r_i\omega$, so

$$L = \sum m_i v_i r_i = \sum m_i (r_i\omega) r_i = \left(\sum m_i r_i^2\right)\omega \qquad \Rightarrow \qquad \boxed{L = I\omega}$$

Compare $p = mv$: moment of inertia for mass, angular velocity for velocity. The faster a body spins, and the larger its moment of inertia (the more mass it has, and the farther from the axis), the larger its angular momentum.

Angular momentum is a **vector**. For a turning body it points along the axis, in the direction given by the **right-hand rule**: curl the fingers of your right hand the way the body turns, and your thumb points along $\vec{L}$.

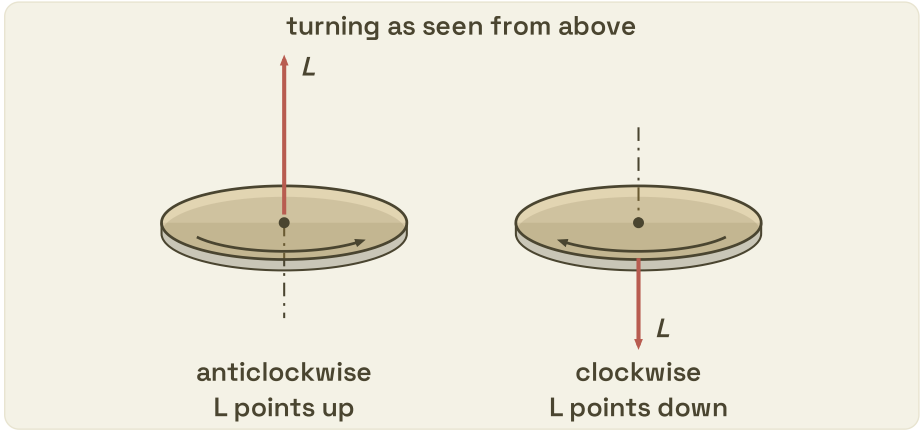


Figure 1.13 *The direction of angular momentum. A disc turning anticlockwise, seen from above, has L pointing up along its axis; turning clockwise, L points down. Curl the fingers of your right hand the way it turns: your thumb gives L.*

Torque changes angular momentum

Newton's second law says that a force changes linear momentum: $F = \dfrac{dp}{dt}$. In the same way, **torque is the rate of change of angular momentum**:

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

acceleration.

When gravity gives a torque, we can treat the body as if all its weight Mg acted at its **center of mass**: the torque of the weight about an axis is Mg times the perpendicular distance from the axis to the vertical line through the center of mass.

Equations of rotational motion

When the angular acceleration is constant, the turning motion follows equations just like Grade 11's equations of motion, with θ in place of s, ω in place of v and α in place of a. If a body starts with angular velocity ω_0 and turns with constant angular acceleration α for a time t:

$$\omega = \omega_0 + \alpha t, \qquad \theta = \omega_0 t + \tfrac{1}{2}\alpha t^2, \qquad \omega^2 = \omega_0^2 + 2\alpha\theta$$

They come the same way: $\alpha = \dfrac{\omega - \omega_0}{t}$ gives the first; the angle turned is the average angular velocity times the time, $\theta = \dfrac{\omega_0 + \omega}{2}t$, which with the first gives the second; and eliminating t gives the third.

WORKED EXAMPLE

Example 1.2: a rod released from horizontal

A uniform rod of mass 2 kg and length 1.2 m is hinged at one end and can turn without friction in a vertical plane. It is held horizontal and then released from rest. Find, just after it is released, a. the torque on it about the hinge, b. its angular acceleration, c. the linear acceleration of its free end. (g = 9.8 m/s²)

Answer

$M = 2$ kg, $L = 1.2$ m. About its end, $I = \frac{1}{3}ML^2$.

a. The weight acts at the center of mass, $L/2 = 0.6$ m from the hinge, at right angles to the horizontal rod:
$\tau = Mg \times \dfrac{L}{2} = 2 \times 9.8 \times 0.6 = \mathbf{11.76\ N\,m}$.

b. $I = \frac{1}{3} \times 2 \times 1.2^2 = 0.96$ kg m², so
$\alpha = \dfrac{\tau}{I} = \dfrac{11.76}{0.96} = \mathbf{12.25\ rad/s^2}$.

c. The free end is L from the hinge: $a = \alpha L = 12.25 \times 1.2 = \mathbf{14.7\ m/s^2}$.
Notice that the free end starts off with an acceleration **greater than g**: $a = \frac{3}{2}g$. A coin resting on the free end would be left behind for an instant.

WORKED EXAMPLE

Example 1.3: stopping the Earth

The Earth has a radius of about 6371 km and turns once every 24 hours. What steady force, pushing along the equator (tangentially), would stop its turning in 1 hour? Treat the Earth as a uniform solid sphere of mass 5.97×10^{24} kg.

Answer

$R = 6371$ km $= 6.371 \times 10^6$ m, $T = 24 \times 3600$ s, stopping time $t = 3600$ s, $M = 5.97 \times 10^{24}$ kg.

Angular velocity now: $\omega_0 = \dfrac{2\pi}{T} = \dfrac{2\pi}{86\,400} = 7.27 \times 10^{-5}$ rad/s.

Angular deceleration (to $\omega = 0$ in 3600 s): $\alpha = \dfrac{\omega_0 - 0}{t} = \dfrac{7.27 \times 10^{-5}}{3600} = 2.02 \times 10^{-8}$ rad/s².

Moment of inertia of a uniform sphere about its axis: $I = \frac{2}{5}MR^2 = 0.4 \times 5.97 \times 10^{24} \times (6.371 \times 10^6)^2 = 9.69 \times 10^{37}$ kg m².

Torque needed: $\tau = I\alpha = 9.69 \times 10^{37} \times 2.02 \times 10^{-8} = 1.96 \times 10^{30}$ N m.

Force at the equator, at right angles to the radius: $F = \dfrac{\tau}{R} = \dfrac{1.96 \times 10^{30}}{6.371 \times 10^6} = \mathbf{3.1 \times 10^{23}\ N}$.

(The real Earth is denser at its center, $I \approx 0.33MR^2$, so the force would be a little less, about 2.5×10^{23} N: about twenty times the weight, on Earth, of all the water in the oceans. Using $F = Ma$ with $a = R\alpha$ treats all the Earth's mass as if it were at the equator, and gives a force two and a half times too large.)

WORKED EXAMPLE

Example 1.4: pulling a string wrapped round a wheel

A string is wrapped round the rim of a wheel of moment of inertia 0.1 kg m² and radius 20 cm. The wheel can turn freely about its axle and is at rest. The string is pulled with a steady force of 20 N. What is the angular velocity of the wheel after 2 s?

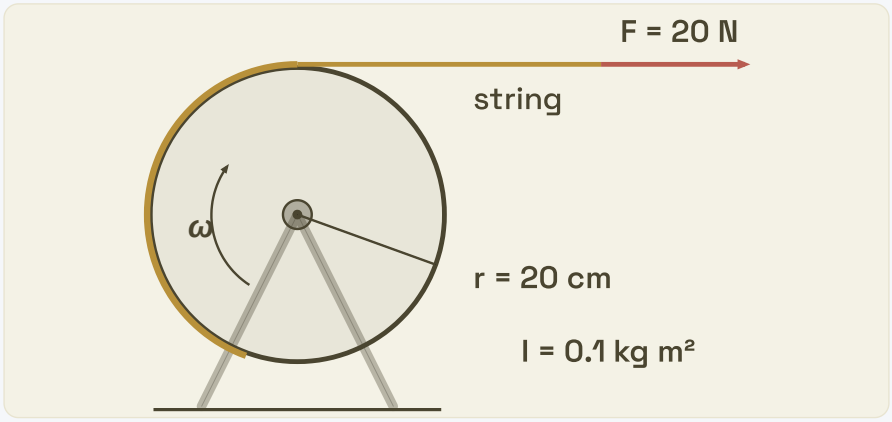


Figure 1.11 *Example 1.4: a string wrapped round a wheel that turns freely on its axle, pulled with a steady force of 20 N along the tangent.*

Answer

The string leaves the rim along a tangent, at right angles to the radius, so the torque is $\tau = Fr = 20 \times 0.2 = 4$ N m.

$\tau = I\alpha \Rightarrow \alpha = \dfrac{\tau}{I} = \dfrac{4}{0.1} = 40$ rad/s².

From rest, $\omega = \omega_0 + \alpha t = 0 + 40 \times 2 = \mathbf{80\ rad/s}$.

YOUR TURN

A rod released at an angle

A uniform rod of mass 2.5 kg and length 1.2 m is hinged at one end and held at 60° to the vertical, in a vertical plane. It is then released from rest. Find, just after release, a. the torque about the hinge due to gravity, b. the angular acceleration, c. the linear acceleration of the free end. (g = 9.8 m/s²)

Answer

a. The weight acts at the centre of mass, 0.6 m from the hinge along the rod. The rod is at 60° to the vertical, so the weight's line is $0.6 \sin 60^\circ$ from the hinge:

$\tau = Mg \times \dfrac{L}{2} \sin 60^\circ = 2.5 \times 9.8 \times 0.6 \times 0.866 = \mathbf{12.73\ N\ m}$.

b. $I = \frac{1}{3}ML^2 = \frac{1}{3} \times 2.5 \times 1.2^2 = 1.2$ kg m², so $\alpha = \dfrac{12.73}{1.2} = \mathbf{10.61\ rad/s^2}$.

c. $a = \alpha L = 10.61 \times 1.2 = \mathbf{12.73\ m/s^2}$.

1.6 Work and power in rotational motion

Work done by a torque

A force does work when it moves its point of action along its own direction. Here is what that gives for turning:

1. A steady force F acts along the tangent at the rim of a wheel of radius r, so its torque about the axle is $\tau = Fr$.
2. The wheel turns through a small angle $d\theta$. The point where the force acts moves along the rim by an arc $ds = r\,d\theta$, in the direction of the force.
3. The work done is $dW = F\,ds = Fr\,d\theta = \tau\,d\theta$.
4. For a steady torque turning the body through an angle θ, adding up the small pieces gives

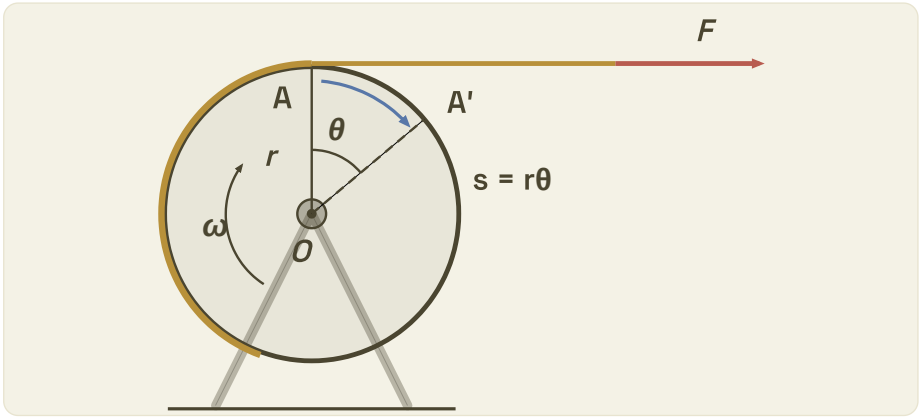


Figure 1.12 *A force F pulls along the tangent at the rim of a wheel of radius r. While the wheel turns through an angle θ, the point where the force acts moves along the arc s = rθ, in the direction of the force.*

with θ in radians. It holds when the torque and the angular displacement are about the same axis. Compare $W = Fs$: torque takes the place of force, and angle the place of distance.

Work and kinetic energy. In straight-line motion, the work done on a body goes into its kinetic energy. In turning, the work done by a torque goes into rotational kinetic energy. If a body starts from rest and a steady torque spins it up to angular velocity ω:

$$W = \tau\theta = I\alpha\theta = I \times \frac{\omega^2}{2} = \tfrac{1}{2}I\omega^2$$

using $\omega^2 = 0 + 2\alpha\theta$. So all the work done is stored as rotational kinetic energy (the work–energy theorem for turning).

Power in rotational motion

Power is the rate of doing work. In straight-line motion, $P = Fv$. In turning, a torque turning a body through a small angle $d\theta$ in a time dt does work $\tau\,d\theta$, so the power is

$$P = \frac{dW}{dt} = \tau\frac{d\theta}{dt} \qquad \Rightarrow \qquad \boxed{P = \tau\omega}$$

The two formulas match: force with torque, velocity with angular velocity. An engine's power is its torque times its angular velocity, which is why car makers quote both.

WORKED EXAMPLE

Example 1.5: an aircraft propeller

An aircraft engine delivers a power of 200 kW to its propeller, which turns at a steady 1500 rpm (revolutions per minute). Find a. the torque the engine exerts on the propeller, b. the work done in one revolution.

Answer

$P = 200$ kW $= 2 \times 10^5$ W; $f = \dfrac{1500}{60} = 25$ rev/s, so $\omega = 2\pi f = 2\pi \times 25 = 50\pi = 157.1$ rad/s.

a. $P = \tau\omega \Rightarrow \tau = \dfrac{P}{\omega} = \dfrac{2 \times 10^5}{50\pi} = \mathbf{1273\ N\ m}$.

b. In one revolution, $\theta = 2\pi$ rad:

$W = \tau\theta = \dfrac{2 \times 10^5}{50\pi} \times 2\pi = \dfrac{2 \times 10^5}{25} = \mathbf{8000\ J}$.

(Check: 25 revolutions a second at 8000 J each is 200 000 J every second, 200 kW.)