Unit 1

Answers: Rotational Dynamics and Oscillatory Motion

Quick checks

1 b Each adds mr^2: the inner $1 \times 0.1^2 = 0.01$ kg m², the outer $1 \times 0.3^2 = 0.09$ kg m², nine times as much. Three times the distance gives nine times the moment of inertia, because the distance is **squared**.

2 a The ring has all its mass at the full radius R, so $I = MR^2$. The disc ($\frac{1}{2}MR^2$) and the solid sphere ($\frac{2}{5}MR^2$) have much of their mass closer to the axis.

3 a The disc has the smaller K^2/R^2 (½ against 1), so less of its energy goes into turning and it moves faster at every point. The mass, the radius and the slope don't change the order: a disc beats a ring on any slope (as long as both roll without slipping).

4 b $\tau = rF\sin\theta = 0.25 \times 40 \times \sin 30^\circ = 0.25 \times 40 \times 0.5 = 5$ N m. 10 N m would be the most it could give, at 90°; 8.7 N m uses cos 30° instead of sin 30°.

5 c $I_1\omega_1 = I_2\omega_2$ with $I_2 = I_1/2$ gives $\omega_2 = 2\omega_1$: 4 turns a second. $KE = \dfrac{L^2}{2I}$, and L is unchanged while I halves, so the kinetic energy doubles; the extra comes from the work her arms do.

6 a $a = \dfrac{g\sin\theta}{1 + K^2/R^2}$: the disc has $K^2/R^2 = \frac{1}{2}$, the ring 1, so the disc has the larger acceleration on any slope. Less of its energy goes into turning.

7 b Equal times for one round means equal angular velocities: $\omega = 2\pi/T$ for both. Then $v = r\omega$, so $\dfrac{v_1}{v_2} = \dfrac{r_1}{r_2}$. The masses don't come in.

8 a Her mass moves nearer the axis, so I falls; with no outside torque, $I\omega$ stays constant, so ω rises.

9 a Ring MR^2; disc and solid cylinder $\frac{1}{2}MR^2$; solid sphere $\frac{2}{5}MR^2$. All the ring's mass is at the full radius.

10 d $I = \sum mr^2$ depends on the mass and where it sits, which the shape decides. Temperature doesn't appear. (Strictly, heating makes a body expand a tiny bit, which moves its mass out and raises I very slightly, but temperature itself is not one of the factors.)

11 b In the air no outside torque acts about her center of mass (gravity acts at it), so $I\omega$ is constant. Tucking brings her mass nearer the axis, so I falls and ω rises.

12 **c** $KE = \frac{1}{2}I\omega^2 = \frac{1}{2}(MK^2)\frac{v^2}{R^2} = \frac{1}{2}Mv^2\frac{K^2}{R^2}$. For wheels of the same shape, K^2/R^2 is the same, so equal masses at equal speeds have equal rotational energy. The smaller wheel turns faster, but its smaller moment of inertia makes up for it exactly.

13 **d** Translational $\frac{1}{2}Mv^2$; rotational $\frac{1}{2}\left(\frac{1}{2}MR^2\right)\frac{v^2}{R^2} = \frac{1}{4}Mv^2$. Total $\frac{3}{4}Mv^2$, of which $\frac{1}{2} \div \frac{3}{4} = \frac{2}{3}$ is translational and $\frac{1}{3}$ rotational.

14 **b** $\tau = rF\sin\theta$, and $\sin 90^\circ = 1$, its largest value: the torque is the greatest it can be for that force and distance, rF.

15 **a** Ring MR^2, hollow sphere $\frac{2}{3}MR^2$, disc $\frac{1}{2}MR^2$, solid sphere $\frac{2}{5}MR^2$. The hollow sphere comes second: its mass is at the radius R, but not all of it at R from the axis (near the poles it is close to the axis).

16 **c** A spinning gyroscope has a large angular momentum along its axis. To change the axis's direction you must change $\vec{L}$, which needs a torque acting for a while; without one, $\vec{L}$, and so the axis, stays fixed. Its moment of inertia and its angular velocity both matter, but only together, through $L = I\omega$.

17 **a** With no outside torque, angular momentum is conserved: it **stays the same**. What changes is the angular velocity: halving the radius makes I ($\propto$ r²) a quarter, so ω becomes four times as large. (If instead the body were kept turning at the same angular velocity, $L = I\omega$ would become a quarter, but that would need an outside torque.)

18 **c** It turns through 2π rad in 1 hour, 3600 s: $\omega = \dfrac{2\pi}{3600} = \dfrac{\pi}{1800}$ rad/s $\approx 1.75 \times 10^{-3}$ rad/s.

14.

a. The **moment of inertia** of a body about an axis is the sum of the products of the mass of each of its particles and the square of its perpendicular distance from the axis: $I = \sum mr^2$. It measures how hard the body is to set turning or to stop. Because each particle counts by the square of its distance, the same mass gives a much larger moment of inertia when it is spread far from the axis than when it is close to it.

b. If no external torque acts on a system, its total angular momentum stays constant: $I\omega = \text{constant}$, or $I_1\omega_1 = I_2\omega_2$. When the gymnast pulls in her arms, her mass comes closer to the axis, so her moment of inertia falls. Nothing outside gives her a torque about her spin axis, so $I\omega$ stays the same, and her angular velocity must rise.

c. i. $I_1\omega_1 = I_2\omega_2 \Rightarrow 5 \times 2 = 2 \times \omega_2 \Rightarrow \omega_2 = \mathbf{5\ rad/s}$.

ii. $KE_1 = \frac{1}{2} \times 5 \times 2^2 = 10$ J; $KE_2 = \frac{1}{2} \times 2 \times 5^2 = 25$ J. Change: $25 - 10 = \mathbf{+15\ J}$ (an increase, from the work her arms did).

iii. $\tau = I\dfrac{\omega_2 - \omega_1}{t} = 5 \times \dfrac{5 - 2}{0.5} = \mathbf{30\ N\ m}$. (In the real spin there is no outside torque at all: her angular momentum stays 10 kg m²/s; she speeds up only by changing her moment of inertia.)

15.

a. **Rolling** is the motion of a round body that turns about its axis while its axis moves along a surface: rotation and translation together. It is **pure rolling** (rolling without slipping) when the point touching the surface is momentarily at rest relative to it, which means the center's speed equals $R\omega$.

b. Let a body of radius R roll without slipping, turning at angular velocity ω. In one complete turn, every point of its rim touches the surface once, so its center moves forward one circumference, $2\pi R$. One turn takes $T = \dfrac{2\pi}{\omega}$. So the speed of the center is $v = \dfrac{2\pi R}{T} = \dfrac{2\pi R}{2\pi/\omega} = R\omega$.

(Or: the bottom point moves forward with the center at v and backward round the center at $R\omega$; for it to be at rest, $v - R\omega = 0$, so $v = R\omega$.)

c. i. The force acts at the end, $\frac{1.2}{2} = 0.6$ m from the center, at right angles to the rod: $\tau = 12 \times 0.6 = \mathbf{7.2\ N\ m}$.

ii. $I = \dfrac{Ml^2}{12} = \dfrac{2.4 \times 1.2^2}{12} = 0.288$ kg m², so $\alpha = \dfrac{7.2}{0.288} = \mathbf{25\ rad/s^2}$.

iii. $a = \dfrac{F}{M} = \dfrac{12}{2.4} = \mathbf{5\ m/s^2}$, the same as if the force acted at the center.

16. a. The **radius of gyration** of a body about an axis is the distance from the axis at which the whole mass of the body, placed at one point, would have the same moment of inertia as the body. It is related to the moment of inertia by $I = MK^2$, or $K = \sqrt{I/M}$.

b. For the same mass M and radius R:

- A **ring** turning about its axis has all its mass at the full radius R, so $I = MR^2$, the largest possible for its size.
- A **disc** has its mass spread from the center to the rim; much of it is close to the axis, so $I = \frac{1}{2}MR^2$, half the ring's.
- A **solid sphere** has even more of its mass close to the axis (the parts near its poles and its core), so $I = \frac{2}{5}MR^2$, the smallest of the three.

The more of a body's mass that lies far from the axis, the larger its moment of inertia.

c. i. In the plane of the incline, the weight's part down the slope is $Mg\sin 30^\circ = 2 \times 9.8 \times 0.5 = 9.8$ N, acting at the center, 0.5 m from the hinge and at right angles to the horizontal rod:

$\tau = 9.8 \times 0.5 =$ **4.9 N m**.

ii. About its end, $I = \dfrac{Ml^2}{3} = \dfrac{2 \times 1^2}{3} = 0.667$ kg m², so

$\alpha = \dfrac{\tau}{I} = \dfrac{4.9}{0.667} =$ **7.35 rad/s²**.

17. i. **Torque** is the turning effect of a force: the product of the force and the perpendicular distance of its line of action from the axis of rotation. SI unit: **N m**. As a vector, $\vec{\tau} = \vec{r} \times \vec{F}$, where $\vec{r}$ is the position vector of the point where the force acts, measured from the axis (or point) of rotation, and $\vec{F}$ is the force. Its size is $rF\sin\theta$, θ being the angle between $\vec{r}$ and $\vec{F}$; its direction is along the axis, perpendicular to both, by the right-hand rule.

ii. $\tau = I\alpha$: the torque on a rigid body about a fixed axis equals its moment of inertia about that axis times its angular acceleration. This is the rotational analog of Newton's second law, $F = ma$, with torque for force, moment of inertia for mass and angular acceleration for acceleration. (In terms of momentum: $\tau = \dfrac{dL}{dt}$, like $F = \dfrac{dp}{dt}$.)

iii. $\alpha = \dfrac{\tau}{I} = \dfrac{1000}{200} = 5$ rad/s². From rest, $\omega = 0 + 5 \times 3 =$ **15 rad/s**.

18. a. **Angular acceleration** is the rate of change of angular velocity, $\alpha = \frac{d\omega}{dt}$; its unit is rad/s². For constant α, with initial angular velocity ω_0:
$\omega = \omega_0 + \alpha t$, $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$, $\omega^2 = \omega_0^2 + 2\alpha\theta$.
b.

	Angular displacement	Angular velocity	Angular acceleration
What it is	the angle turned through about the axis	the rate of change of angular displacement	the rate of change of angular velocity
Formula	θ (= arc ÷ radius)	$\omega = \frac{d\theta}{dt}$	$\alpha = \frac{d\omega}{dt}$
SI unit	rad	rad/s	rad/s²
Linear partner	displacement	velocity	acceleration

c. i. $\alpha = \frac{\omega - \omega_0}{t} = \frac{20 - 0}{5} =$ **4 rad/s²**.
ii. $\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = 0 + \frac{1}{2} \times 4 \times 5^2 =$ **50 rad** (about 8 turns).

19. a. The **center of mass** of a body is the point at which its whole mass can be taken to be concentrated, for describing its motion as a whole: a force through it moves the body without turning it. For a body turning about a fixed axis, if the center of mass lies **on** the axis (the masses balanced round it), its weight gives no torque about the axis and the axle feels no wobbling force as it spins, so it turns smoothly and steadily. If the center of mass is off the axis, the body is out of balance: gravity tends to turn it to bring the center of mass lowest, and at speed the shaking forces on the axle can damage it. This is why wheels are balanced with small weights at a garage.

b. **Rotational kinetic energy** is the kinetic energy a body has because it turns: the sum of the kinetic energies of all its particles going round their circles, $KE = \frac{1}{2}I\omega^2$. **Translational kinetic energy**, $\frac{1}{2}mv^2$, is the energy of a body's motion as a whole from place to place. A wheel spinning on a fixed axle has only rotational kinetic energy; a box sliding along has only translational; a rolling wheel has both. Rotational KE depends on how the mass is spread (through I), translational KE only on the total mass.

c. $I = \sum mr^2 = 1 \times 0.2^2 + 2 \times 0.1^2 = 0.04 + 0.02 =$ **0.06 kg m²**.

20.

a. Mass measures how hard it is to change a body's straight-line motion ($F = ma$); moment of inertia measures how hard it is to change its turning ($\tau = I\alpha$). It takes mass's place in every turning formula: $L = I\omega$ for $p = mv$, $\frac{1}{2}I\omega^2$ for $\frac{1}{2}mv^2$. The one difference is that it depends on where the mass is, not only how much.

b. Take a uniform rod of mass M and length l, the origin at its center, the rod along the x-axis from $-l/2$ to $+l/2$. A piece of length dx at distance x has mass $dm = \dfrac{M}{l}dx$ and moment of inertia $dI = x^2\,dm = \dfrac{M}{l}x^2\,dx$. So

$$I = \int_{-l/2}^{l/2} \frac{M}{l}x^2\,dx = \frac{M}{l}\left[\frac{x^3}{3}\right]_{-l/2}^{l/2} = \frac{M}{l} \times \frac{2}{3} \times \frac{l^3}{8} = \frac{Ml^2}{12}.$$

c. $I = \dfrac{5 \times 1^2}{12} = \mathbf{0.417\ kg\ m^2}$.
$KE = \frac{1}{2}I\omega^2 = \frac{1}{2} \times 0.4167 \times 12^2 = \mathbf{30\ J}$.

21.

a. $I = \dfrac{ML^2}{12} + m_1\left(\dfrac{L}{2}\right)^2 + m_2\left(\dfrac{L}{2}\right)^2$: the first term is the rod's own moment of inertia about its center, the other two are the point masses, each its mass times the square of its distance from the pivot.

b. It **decreases**. The moved mass is now nearer the axis, so its mr^2 is smaller; the other terms are unchanged.

c. Rod: $\dfrac{3 \times 1.2^2}{12} = 0.36$ kg m². Masses: $2 \times 1 \times 0.6^2 = 0.72$ kg m². Total: **1.08 kg m²**.

22.

a. The **angular momentum** of a body turning about an axis is the product of its moment of inertia about the axis and its angular velocity, $L = I\omega$ (for a particle, the moment of its linear momentum, $\vec{L} = \vec{r} \times \vec{p}$). SI unit: **kg m²/s**.

b. If no external torque acts on a system, its total angular momentum stays constant: $I_1\omega_1 = I_2\omega_2$. Since $\tau = dL/dt$, zero torque means L cannot change; so if the moment of inertia changes, the angular velocity changes the opposite way. Example: a diver tucks in to somersault faster; or a planet moves faster when it is nearer the Sun.

c. $\omega_2 = \dfrac{I_1\omega_1}{I_2} = \dfrac{3 \times 4}{1.5} = \mathbf{8\ rad/s}$. Her angular momentum is $3 \times 4 = 12$ kg m²/s before and $1.5 \times 8 = 12$ kg m²/s after: it is conserved, because no outside torque acted during the change. She spins faster only because her moment of inertia fell.

23.

a. Let a force F act along the tangent at the rim of a wheel of radius r, so its torque is $\tau = Fr$. When the wheel turns through a small angle $d\theta$, the point where the force acts moves along the arc $ds = r\,d\theta$, in the direction of the force. The work done is $dW = F\,ds = Fr\,d\theta = \tau\,d\theta$. For a steady torque turning the wheel through θ, $W = \tau\theta$ (θ in radians).

b. $W = \tau\theta = 8 \times 5 = \mathbf{40\ J}$. Average power $P = \dfrac{W}{t} = \dfrac{40}{2} = \mathbf{20\ W}$.

24.

a. The **principle of conservation of angular momentum**: if no external torque acts on a system, its total angular momentum stays constant ($I_1\omega_1 = I_2\omega_2$). Pulling her arms in reduces her moment of inertia, so her angular velocity rises.

b. Let a rigid body turn with angular velocity ω about a fixed axis, its particles $m_1, m_2, \ldots$ at distances $r_1, r_2, \ldots$ from the axis. A particle at r_i moves round its circle at $v_i = r_i\omega$, so its angular momentum about the axis is $m_i v_i r_i = m_i r_i^2 \omega$. Adding them, with the same ω for all: $L = \left(\sum m_i r_i^2\right)\omega = I\omega$.

c. $\omega_2 = \dfrac{I_1\omega_1}{I_2} = \dfrac{4 \times 2}{2} = \mathbf{4\ rad/s}$.

25.

a. Moment of inertia does for turning what mass does for straight-line motion: it is the body's resistance to a change in its turning. For a given torque, a large moment of inertia means a small angular acceleration ($\alpha = \tau/I$); it also sets a turning body's angular momentum ($I\omega$) and kinetic energy ($\frac{1}{2}I\omega^2$).

b. $I = \dfrac{ML^2}{12}$.

c. If no external torque acts, the total angular momentum of a system stays constant, $I_1\omega_1 = I_2\omega_2$. Example: a diver tucks in to turn faster in a somersault.

d. $I = \dfrac{6 \times 2^2}{12} + 2 \times 0.5^2 + 3 \times 0.8^2 = 2 + 0.5 + 1.92 = \mathbf{4.42\ kg\ m^2}$.

i. $\alpha = \dfrac{\tau}{I} = \dfrac{12}{4.42} = \mathbf{2.71\ rad/s^2}$.

ii. From rest, $\omega^2 = 0 + 2\alpha\theta = 2 \times 2.715 \times \pi = 17.06$, so $\omega = \mathbf{4.13\ rad/s}$.

26.

a. **Angular momentum** of a body about an axis is its moment of inertia times its angular velocity, $L = I\omega$: the turning equivalent of linear momentum. SI unit: **kg m²/s**.

b. If no external torque acts on a system, its total angular momentum remains constant.

c. Pulling her arms in brings mass closer to the spin axis, so her moment of inertia falls from I_1 to I_2. There is no outside torque about the vertical axis (the ice is nearly frictionless), so $I\omega$ can't change: with a smaller I, ω must be larger.

d. i. $\omega_2 = \dfrac{I_1\omega_1}{I_2} = \dfrac{5 \times 2}{2} = \mathbf{5\ rad/s}$.

ii. $KE_1 = \frac{1}{2} \times 5 \times 2^2 = 10$ J, $KE_2 = \frac{1}{2} \times 2 \times 5^2 = 25$ J: an increase of **15 J**, supplied by the work her arms do pulling inward.