Unit 2 · Periodic Motion

STUDY NOTES

WHAT NEB'S GRADE 12 EXAM ASKS

- **CDC's grid (2078):** Mechanics (Units 1–3 together): 13 of 75 marks
- **The paper:** 11 MCQs × 1 · 8 short × 5 · 3 long × 8; 3 hours; 1 mark per expected point
- **CDC's model paper:** a long question (8 marks): $F = -Ax$ gives SHM; prove $v = \omega\sqrt{r^2 - y^2}$; a building's greatest speed in S-waves ($v_{\max} = r\omega$); why tall buildings suffer from low-frequency waves (resonance)
- **MCQ (1 mark)**
  - amplitude of $3\sin\omega t + 4\cos\omega t$
  - angle between v and a
  - time from r to $r/2$
  - pendulum: length tripled; on the Moon
  - two girls on one swing
  - spring: mass or k changed
  - springs in series
  - when resonance happens; examples
  - torsion: $\omega = \sqrt{\kappa/I}$
- **Short and long (5 or 8 marks, in parts)**
  - define SHM; derive $a = -\omega^2 y$, the energy, T of spring, pendulum, torsion pendulum
  - equilibrium position; vertical spring's force; $T = 2\pi\sqrt{h/g}$
  - pendulum: free-body diagram, length from a half swing
  - damping and its kinds; sketch a damped graph; in oil
  - resonance: condition, phase, the amplitude–frequency graph
  - numericals: energy, KE at y, springs in parallel, y, v, a at a time, amplitude falling 5% a cycle

Key terms

Periodic motion: repeats itself at regular intervals; the time of one repeat is the **period**.

Oscillation: periodic motion back and forth about a fixed point, along the same path (a fan's turning is periodic, not an oscillation).

SHM: restoring force directly proportional to the displacement from the mean position and always towards it; or the **acceleration** so.

Amplitude: greatest displacement from the mean position (the reference circle's radius).

Period T: time for one complete oscillation; **frequency** f: oscillations per second (Hz).

Phase: $\omega t + \phi$, where the particle is in its cycle and which way it moves; **epoch** ϕ: the phase at $t = 0$.

Phase difference: how far one oscillation leads or lags another; in phase 0, 2π…; antiphase π.

Simple pendulum: heavy point bob on a light inextensible string; **effective length** = string + bob's radius.

Seconds pendulum: period **2 s** (1 s each way); 0.993 m.

Angular SHM: restoring **torque** proportional to the **angular** displacement.

Natural frequency f_0: frequency of a free oscillation, set by the body alone.

Damping: loss of an oscillation's energy to friction or air; amplitude falls.

Forced oscillation: at the **driving** frequency of a periodic outside force.

Resonance: driving frequency = natural frequency → largest amplitude.

Formulas

$F = -kx,\ a = -\omega^2 x$	$\omega^2 = k/m$ **Use:** test for SHM
$y = r\sin(\omega t + \phi)$	r amplitude **Use:** displacement
$v = \omega\sqrt{r^2 - y^2}$	$v_{\max} = r\omega$ at the mean **Use:** velocity
$a = -\omega^2 y$	$a_{\max} = \omega^2 r$ at the extremes **Use:** acceleration
$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$	any SHM **Use:** period
$\omega = 2\pi f = 2\pi/T$	rad/s **Use:** angular frequency

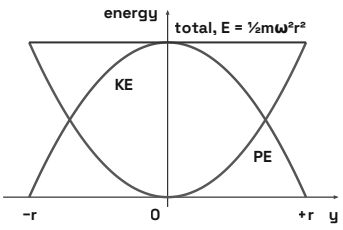
$E = \frac{1}{2}m\omega^2 r^2$	constant; $PE = \frac{1}{2}m\omega^2 y^2$ **Use:** energy
$T = 2\pi\sqrt{m/k} = 2\pi\sqrt{x/g}$	x stretch at rest **Use:** mass on a spring
$k = k_1 + k_2$; $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$	parallel; series **Use:** springs joined
$T = 2\pi\sqrt{l/g}$	small swings, $\theta < 5°$ **Use:** simple pendulum
$T^2 = \frac{4\pi^2}{g}\,l$	slope $4\pi^2/g$ **Use:** measuring g
$T = 2\pi\sqrt{I/\kappa}$	κ N m/rad **Use:** torsion pendulum

SHM and the reference circle

- point P round a circle (radius r, angular velocity ω); its shadow N on a diameter moves in SHM
- $y = r\sin\omega t \to v = dy/dt = r\omega\cos\omega t = \omega\sqrt{r^2 - y^2}$ $\to a = dv/dt = -r\omega^2\sin\omega t = -\omega^2 y$
- speed greatest where $a = 0$ (mean); a greatest where it stops (extremes)
- $\phi = 0$: starts at the mean, moving up; $\phi = \pi/2$: at the top, $y = r\cos\omega t$
- v and a are a quarter cycle apart (cos and $-$ sin); a always towards the mean
- $3\sin\omega t + 4\cos\omega t = 5\sin(\omega t + \phi)$: amplitude $\sqrt{3^2 + 4^2}$, not $3 + 4$
- extreme $\to r/2$: $\cos\omega t = \frac{1}{2} \to t = T/6$ (slow near the extreme)
- a–x graph: a straight line through the origin, slope $-\omega^2$

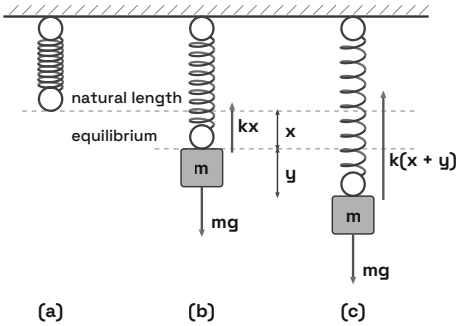
Energy

- $F = -m\omega^2 y = -ky$; work against it from 0 to y: $PE = \int_0^y ky\,dy = \frac{1}{2}ky^2 = \frac{1}{2}m\omega^2 y^2$
- $KE = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(r^2 - y^2)$
- sum: $E = \frac{1}{2}m\omega^2 r^2$, the y cancels: same everywhere; $E \propto r^2$, $\propto f^2$
- mean: all KE; extremes: all PE; KE = PE at $y = r/\sqrt{2}$



PE (parabola), KE (upside down) and total E against displacement.

Mass on a vertical spring

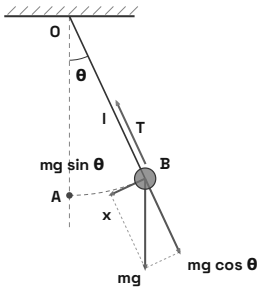


Spring (a) unstretched; (b) at rest, $kx = mg$; (c) pulled down y: spring pulls $k(x + y)$.

1. at rest: $kx = mg$
2. pulled a further y: $F = mg - k(x + y) = -ky$
3. $a = -\frac{k}{m}y \to$ SHM, $\omega = \sqrt{k/m}$, $T = 2\pi\sqrt{m/k}$

- oscillates about the **equilibrium** position, not the unstretched length
- T independent of amplitude and g (same on the Moon); gravity only shifts the mean by mg/k
- spring's own mass neglected
- two equal springs: parallel $2k$ ($T \div \sqrt{2}$), series $k/2$ ($T \times \sqrt{2}$)

Simple pendulum



Weight split: $mg\cos\theta$ along the string (balanced by the tension), $mg\sin\theta$ along the arc (restoring).

1. restoring force $F = -mg\sin\theta$
2. small θ (radians): $\sin\theta \approx \theta = x/l \to F = -\frac{mg}{l}x$
3. $a = -\frac{g}{l}x \to$ SHM, $\omega = \sqrt{g/l}$, $T = 2\pi\sqrt{l/g}$

- tension slightly more than $mg\cos\theta$: it also gives the centripetal force
- $T \propto \sqrt{l}$ (4 × length, 2 × period); $T \propto 1/\sqrt{g}$ (Moon: $\times\sqrt{6} \approx 2.45$)
- **not** on the bob's mass; **not** on the amplitude (small swings): **isochronism** (Galileo; Huygens' clock, 1656–57)
- T^2–l graph: a straight line through the origin; $g = 4\pi^2/\text{slope}$ (best-fit line averages errors)

- seconds pendulum: $l = gT^2/4\pi^2 = 9.8/\pi^2 = 0.993$ m

Why a real pendulum isn't simple

1. only small swings: $15^\circ \to 0.4$ % longer, $30^\circ \to 1.7$ %
2. g changes from place to place
3. heat lengthens it $\to$ runs slow
4. air drag and pivot friction: amplitude dies
5. vibrations, wind
6. a clock's escapement must keep pushing it

Angular SHM: torsion pendulum

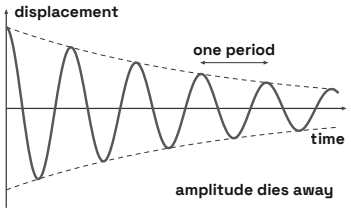
- disc hung at its center by a wire, turned through θ and let go

1. twisted wire: $\tau = -\kappa\theta$
2. $I\alpha = -\kappa\theta \to \alpha = -\frac{\kappa}{I}\theta$
3. angular SHM: $\omega = \sqrt{\kappa/I}$, $T = 2\pi\sqrt{I/\kappa}$

- $4 \times I \to 2 \times$ period; needs no small-angle rule; a watch's balance wheel and hairspring

Free, damped, forced

- **free**: at its natural frequency; ideal $\to$ same amplitude forever
- **damped**: amplitude falls by the same **fraction** each cycle (exponential); each cycle a little **longer**

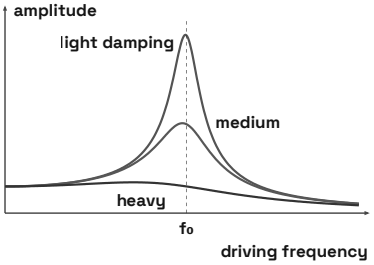


Damped oscillation: peaks on a dying exponential curve.

Kinds of damping

1. **light** (underdamped): still oscillates, smaller each swing (pendulum in air, tuning fork)
2. **critical**: back to the mean in the **shortest** time **without** overshooting (shock absorbers, meters)
3. **heavy** (overdamped): no oscillation, creeps back (syrup, a door closer)

- damped examples: pendulum in air; shock absorbers; tuning fork; slow-closing door; spring in water
- **forced**: oscillates at the driving frequency; each push makes up the energy lost



Amplitude against driving frequency: peak at f_0; lighter damping $\to$ higher, sharper.

- at resonance the displacement **lags the force by $\pi/2$**; velocity in step with the force $\to$ energy in fastest
- slow driving: in step; fast: half a cycle behind

Resonance examples

1. swing pushed in time
2. tuning fork on a resonance box; guitar, sarangi bodies
3. glass broken by a loud note
4. tuning a radio
5. buildings in earthquakes; tuned mass dampers
6. soldiers break step on a bridge

- Tacoma Narrows (1940): really aeroelastic flutter in steady wind; exams accept it as resonance

COMMON MISTAKES

✗ seconds pendulum has $T = 1$ s
✓ $T = \mathbf{2}$ **s** (1 s each way); 0.993 m.

✗ a spring's period depends on g
✓ $T = 2\pi\sqrt{m/k}$ only; g only moves the mean position.

✗ the tension provides $mg\cos\theta$
✓ it **balances** it, and is a little larger.

✗ $a = r\omega^2 \sin\omega t$
✓ $a = -\omega^2 y$: the minus means towards the mean.

✗ heavier bob, slower pendulum
✓ mass doesn't matter.

✗ critical damping = no damping
✓ just enough to return fastest **without** overshooting.

✗ at resonance displacement and force are in step
✓ displacement **lags** by $\pi/2$.