Unit 1 · Rotational Dynamics and Oscillatory Motion

STUDY NOTES

WHAT NEB'S GRADE 12 EXAM ASKS

- **CDC's grid (2078):** Mechanics (Units 1–3 together): 13 of 75 marks
- **The paper:** 11 MCQs × 1 · 8 short × 5 · 3 long × 8; 3 hours; 1 mark per expected point
- **CDC's model paper:** MCQ: the radius of gyration's formula ($K^2 = I/M$); short: define moment of inertia, why the Earth moves faster near the Sun, what a suddenly shrinking Earth does to the day
- **MCQ (1 mark)**
  - disc vs ring down a slope
  - skater or diver pulls in: $I \downarrow, \omega \uparrow$
  - largest I of ring, disc, spheres
  - what doesn't affect I
  - $v_1 : v_2$ in equal times
  - rolling KE: both kinds
  - torque at $\theta = 90°$
  - radius halved: L unchanged
  - minute hand's ω
  - why a gyroscope stays steady
- **Short and long (5 or 8 marks, in parts)**
  - define I, K, torque, angular momentum, rolling; their units
  - derive: rod's I (center, end); $KE = \frac{1}{2}I\omega^2$; $v = R\omega$; $\tau = I\alpha$; $W = \tau\theta$; $L = I\omega$
  - state and explain conservation of L, with examples
  - numericals: skater's new ω and KE change; rod pushed at one end; hinged rod's α; wheel under a torque; I of a rod with masses

Key terms

Rigid body: distance between any two of its particles never changes, whatever the forces.

Translation: every point moves along parallel paths by the same displacement; **rotation**: every point moves on a circle about the axis.

Moment of inertia I: $\sum mr^2$ about an axis; the turning equivalent of mass; scalar, kg m^2; depends on the mass **and** its distance from the axis (and so on the axis).

Radius of gyration K: distance from the axis at which the whole mass, at one point, would have the same I; m.

Torque τ: force × perpendicular distance from the axis to its line of action; $\vec{\tau} = \vec{r} \times \vec{F}$; N m.

Rolling: a round body turns about its axis while the axis moves forward; **pure rolling**: the point touching the ground is at rest.

Angular momentum L: moment of linear momentum, $\vec{L} = \vec{r} \times \vec{p}$; kg m^2/s (= J s).

Conservation of angular momentum: with no external torque, the total angular momentum of a system stays constant.

Formulas

$I = \sum m_i r_i^2 = \int r^2\, dm$	$r \perp$ distance from the axis **Use:** moment of inertia
$I = MK^2$	$K = \sqrt{I/M}$ **Use:** radius of gyration
$K = \sqrt{(r_1^2 + \cdots + r_n^2)/n}$	rms distance **Use:** n equal particles
$KE_{\text{rot}} = \frac{1}{2}I\omega^2$	ω in rad/s **Use:** spinning body
$v = R\omega$	R radius **Use:** pure rolling
$KE = \frac{1}{2}Mv^2\left(1 + \frac{K^2}{R^2}\right)$	moving + turning **Use:** rolling body
$v = \sqrt{\frac{2gh}{1+K^2/R^2}}$	only the shape counts **Use:** rolling down a slope

$\tau = rF\sin\theta$	θ between $\vec{r}, \vec{F}$ **Use:** torque
$\tau = I\alpha$	turning $F = ma$ **Use:** angular acceleration
$\omega = \omega_0 + \alpha t$	constant α **Use:** rotation
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	θ in rad **Use:** rotation
$\omega^2 = \omega_0^2 + 2\alpha\theta$	no t **Use:** rotation
$W = \tau\theta,\ P = \tau\omega$	θ in radians **Use:** work, power
$L = I\omega,\ \tau = \frac{dL}{dt}$	kg m^2/s **Use:** angular momentum
$I_1\omega_1 = I_2\omega_2$	no external torque **Use:** conservation

- slope: $a = \dfrac{g\sin\theta}{1 + K^2/R^2}$; no M, no R

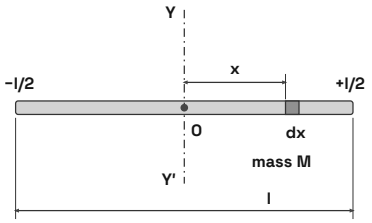
Moment of inertia

- more mass **farther** from the axis → larger I; a particle twice as far counts **4 ×**
- same body, different axis → different I: solid sphere $\frac{2}{5}MR^2$ about a diameter, $\frac{7}{5}MR^2$ about a tangent; always say **about which axis**
- K depends on the axis and the spread of mass, **not** on how much mass
- I is numerically **twice the KE** at $\omega = 1$ rad/s

Body (mass M)	Axis	I	K^2/R^2
ring, hollow cylinder	own axis	MR^2	1
hollow sphere	diameter	$\frac{2}{3}MR^2$	2/3
solid disc, cylinder	own axis	$\frac{1}{2}MR^2$	1/2
solid sphere	diameter	$\frac{2}{5}MR^2$	2/5
rod, length l	center, ⊥	$\frac{1}{12}Ml^2$	
rod, length l	one end, ⊥	$\frac{1}{3}Ml^2$	

- same M, R: **ring > hollow sphere > disc > solid sphere**

Rod about its center: derivation



Rod of mass M, length l, axis YY′ through O; a piece dx at x has dm = (M/l) dx.

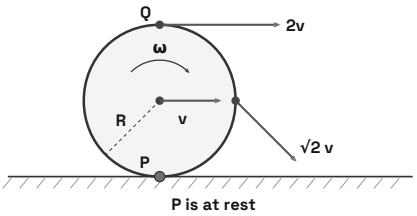
1. mass per length M/l; piece $dm = \frac{M}{l}dx$ at x
2. $dI = \frac{M}{l}x^2\,dx$
3. $I = \int_{-l/2}^{l/2}\frac{M}{l}x^2\,dx = \frac{M}{l}\left[\frac{x^3}{3}\right]_{-l/2}^{l/2} = \frac{Ml^2}{12}$; $K = \frac{l}{2\sqrt{3}} \approx 0.29\,l$

- **about one end**: same steps, $\int_0^l \to I = \frac{Ml^2}{3}$, $K = l/\sqrt{3} \approx 0.58\,l$; **4 ×** the center's
- check: $I = I_{\text{cm}} + Md^2 \to \frac{Ml^2}{12} + M\frac{l^2}{4} = \frac{Ml^2}{3}$

Rotational KE and rolling

KE_{rot}: derivation

1. rigid → every particle has the same ω; $v_i = r_i\omega$
2. $KE_i = \frac{1}{2}m_i r_i^2\omega^2$
3. sum: $\frac{1}{2}\omega^2\sum m_i r_i^2 = \frac{1}{2}I\omega^2$ (I for m, ω for v)



Pure rolling: center at $v = R\omega$; bottom point P at rest, top Q at $2v$.

- $v = R\omega$: one turn lays $2\pi R$ on the ground in $T = 2\pi/\omega \to v = 2\pi R/T = R\omega$; or the bottom point: $v - R\omega = 0$
- $v > R\omega$: skids; $v < R\omega$: spins and slips (wheels on mud)
- solid cylinder rolling: ⅔ of KE translational, ⅓ rotational
- slope race (from rest, height h): solid sphere $1.20\sqrt{gh}$ first → disc 1.15 → hollow sphere 1.10 → ring $\sqrt{gh}$ last; sliding without friction $1.41\sqrt{gh}$
- bigger K^2/R^2 → more energy into turning → slower

Torque and $\tau = I\alpha$

- τ greatest at $\theta = 90°$ ($\tau = rF$); **zero** at 0° or 180° (force along $\vec{r}$)

- direction: along the axis, right-hand rule (fingers the way it turns, thumb = $\vec{\tau}$)
- gravity's torque: weight Mg at the **center of mass** × perpendicular distance

$\tau = I\alpha$: derivation

1. rigid → same α; tangential $a_i = r_i\alpha$
2. $F_i = m_i r_i \alpha$, at right angles to r_i
3. $\tau_i = F_i r_i = m_i r_i^2 \alpha$
4. sum (internal torques cancel in pairs): $\tau = (\sum m_i r_i^2)\alpha = I\alpha$

- equations of rotation: $\alpha = (\omega - \omega_0)/t$; θ = average ω × t; eliminate t for the third

Work, power, angular momentum

- $W = \tau\theta$: tangential F at the rim, arc $ds = r\,d\theta$ → $dW = F r\,d\theta = \tau\,d\theta$ → $W = \tau\theta$
- from rest: $W = I\alpha\theta = \frac{1}{2}I\omega^2$ (work–energy theorem for turning)
- $P = dW/dt = \tau\,d\theta/dt = \tau\omega$ (engine: torque × ω)
- $L = I\omega$: particle on a circle $L = mvr$; sum $m_i(r_i\omega)r_i = (\sum m_i r_i^2)\omega$
- L is a vector along the axis (right-hand rule)
- $\tau = dL/dt = I\,d\omega/dt = I\alpha$; torque can change the spin rate, the axis's direction, or both
- $\tau = 0 \rightarrow dL/dt = 0 \rightarrow I\omega$ constant → $I_1\omega_1 = I_2\omega_2$

Conservation of L: examples

1. skater, diver: arms and legs in → $I \downarrow \rightarrow \omega \uparrow$; out → slower
2. planet: Sun's pull gives no torque → mvr constant → faster near the Sun (Earth 30.3 km/s in January, 29.3 km/s in July; Kepler's second law)
3. tornado, whirlpool: air or water moving in goes round faster
4. melting polar ice → water toward the equator → $I \uparrow$ → day longer (about 1.3 ms per century since 2000)
5. gyroscope, top, spinning wheel: keeps its axis (changing L needs a torque)
6. turning chair with weights: hands in → faster, out → slower

- skater, I halved: ω doubles, KE ($L^2/2I$) doubles: the extra comes from the work her arms do
- light bicycle rim: small I for its size → spins up quickly; thin spokes in tension hold it

Linear ↔ rotational

- same period (equal times) → same ω → $v_1 : v_2 = r_1 : r_2$
- minute hand: $\omega = 2\pi/3600 \approx 1.75 \times 10^{-3}$ rad/s

Linear	Rotational
s, v, a	θ, ω ($v = r\omega$), α ($a = r\alpha$)
mass m	$I = \sum mr^2$
$p = mv$	$L = I\omega$
$F = ma = dp/dt$	$\tau = I\alpha = dL/dt$
$W = Fs$, $P = Fv$	$W = \tau\theta$, $P = \tau\omega$
$\frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$
no external force: p conserved	no external torque: L conserved

COMMON MISTAKES

✗ $K = (r_1^2 + \cdots + r_n^2)/n$
✓ $K = \sqrt{(r_1^2 + \cdots + r_n^2)/n}$, for **equal** masses only.

✗ I = total mass × size^2 for any body
✓ only for a ring or a point mass; others need $\sum mr^2$.

✗ a disc and a ring tie down a slope
✓ the disc wins (smaller K^2/R^2); mass and radius don't matter.

✗ radius halved, so L changes
✓ no outside torque: L stays; ω becomes 4 ×.

✗ a bicycle wheel is a solid disc
✓ nearly a ring (mr^2): its mass is in the rim and tyre.

✗ skater pulls in: KE stays constant
✓ L stays; KE **rises**.

✗ θ in degrees in $W = \tau\theta$
✓ in **radians**.