Unit 16 · Linear Programming Problems

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 20

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 How far is a shadow price good for?

HA · 5

ANSWER

Tests: a shadow price holds only while the same corner (the same basic variables) stays optimal; the tempting answer multiplies it by any increase.

a. With $b = 10$: the corners are O, (4, 0), (0, 8) (where $2x + y = 8$ meets the y-axis; $x + y \leq 10$ no longer binds). $Z(4, 0) = 20$, $Z(0, 8) = 32$. **Maximum 32 at (0, 8)**, not 38. (The simplex: x in, s_2 out, then y in, x out; final $Z = 32$, $y = 8$, $s_1 = 2$.)

b. Subtracting: $x = 8 - b$, $y = 2b - 8$. Both ≥ 0 for $\mathbf{4} \leq b \leq \mathbf{8}$. There $Z = 5(8 - b) + 4(2b - 8) = 8 + 3b$: each unit of b adds 3, the shadow price.

c. The 3 is the rate only while x and y are both basic. At $b = 8$, x reaches 0 ($Z = 32$); past that, more of the first resource can't be used at all ($s_1 > 0$), so its value drops to 0. From 5 to 10, only 3 units are worth 3 each: $23 + 9 = 32$.

MARKING SCHEME

✓ Maximum 32 at (0, 8) for $b = 10$.
✓ $x = 8 - b$, $y = 2b - 8$.
✓ A corner for $4 \leq b \leq 8$, where $Z = 8 + 3b$.
✓ At $b = 8$ the corner reaches the axis ($x = 0$).
✓ Past it the first resource is left over: its shadow price falls to 0, so only 3 units add 3 each.

2 Changing a price

HA · 4

ANSWER

Tests: reading the optimum from the slopes at a corner; a 0 under a non-basic variable in the final row means a tie (more than one optimum); the tempting answer is that the method's point is the only best plan.

a. The lines $Z =$ constant have slope $-\frac{c}{4}$; at (3, 2) the edges have slopes -1 ($x + y = 5$) and -2 ($2x + y = 8$). (3, 2) is best while $-2 \leq -\frac{c}{4} \leq -1$: $\mathbf{4} \leq c \leq \mathbf{8}$.

b. Bringing s_2 in would change Z by 0: there is another corner as good. $c = 4$: $Z = 4(x + y)$, and every point of the edge $x + y = 5$ between (3, 2) and (0, 5) gives $Z = 20$. Another best plan: **(0, 5)**, $Z = 20$ (or any point of that edge).

MARKING SCHEME

✓ The objective's slope $-\frac{c}{4}$ between the edges' slopes -2 and -1.
✓ $4 \leq c \leq 8$.
✓ A 0 under a non-basic variable: another optimum.
✓ (0, 5) (or a point of the edge), $Z = 20$.

3 Three products at the pottery

ANSWER

Tests: setting up and pivoting a problem with three variables, which no graph shows; that the first variable to enter need not stay; reading a product left out, and its reduced cost.

a. Maximize $Z = 20x + 30y + 25z$ subject to $x + 2y + z \leq 40$, $2x + y + 3z \leq 60$, $x, y, z \geq 0$.

b. First tableau: rows (1, 2, 1, 1, 0 | 40), (2, 1, 3, 0, 1 | 60), bottom (−20, −30, −25, 0, 0 | 0).

y enters (-30); ratios 20 and 60: s_1 leaves. $R_1 \to \frac{1}{2}R_1$, $R_2 \to R_2 - R_1$, $R_3 \to R_3 + 30R_1$:

y: $\left(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, 0 \mid 20\right)$; s_2: $\left(\frac{3}{2}, 0, \frac{5}{2}, -\frac{1}{2}, 1 \mid 40\right)$; bottom $(-5, 0, -10, 15, 0 \mid 600)$.

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z enters (-10); ratios $\frac{20}{1/2} = 40$ and $\frac{40}{5/2} = 16$: s_2 leaves. $R_2 \to \frac{2}{5}R_2$, $R_1 \to R_1 - \frac{1}{2}R_2$, $R_3 \to R_3 + 10R_2$: y: $\left(\frac{1}{5}, 1, 0, \frac{3}{5}, -\frac{1}{5} \mid 12\right)$; z: $\left(\frac{3}{5}, 0, 1, -\frac{1}{5}, \frac{2}{5} \mid 16\right)$; bottom $(1, 0, 0, 13, 4 \mid 760)$. **A · 6**

No negative entry: **12 bowls and 16 pots, no cups, Rs 760 a day**. Check: clay $24 + 16 = 40$ ✓, painting $12 + 48 = 60$ ✓.

c. The 1 under x: each cup made would lower the profit by Rs 1 (the clay and painting it uses are worth $13 \times 1 + 4 \times 2 =$ Rs 21 in bowls and pots, and it earns only Rs 20). Its profit must rise **above Rs 21**, by more than Rs 1.

MARKING SCHEME

- ✓ The objective and the two constraints.
- ✓ First tableau, y in, s_1 out.
- ✓ Second tableau, $Z = 600$.
- ✓ z in, s_2 out ($16 < 40$).
- ✓ 12 bowls, 16 pots, no cups, Rs 760.
- ✓ The 1 under x: a cup's resources are worth Rs 21, so its profit must pass Rs 21.

4 A minimum by the same tableau

ANSWER **A · 5**

Tests: a minimum as the maximum of $-C$; that a variable which "costs" on its own can still belong to the best plan, because it loosens a constraint (the tempting answer: $x = 0$).

a. C is least exactly where $-C$ is greatest, and $\min C = -\max(-C)$. Maximize $Z = -x + 3y$: bottom row $(1, -3, 0, 0 \mid 0)$.

y enters; ratios $\frac{6}{1} = 6$, $\frac{6}{2} = 3$: s_2 leaves. Then s_1: $\left(\frac{3}{2}, 0, 1, -\frac{1}{2} \mid 3\right)$; y: $\left(-\frac{1}{2}, 1, 0, \frac{1}{2} \mid 3\right)$; bottom $\left(-\frac{1}{2}, 0, 0, \frac{3}{2} \mid 9\right)$.

x enters ($-\frac{1}{2}$); only row 1 is positive: s_1 leaves. x: $\left(1, 0, \frac{2}{3}, -\frac{1}{3} \mid 2\right)$; y: $\left(0, 1, \frac{1}{3}, \frac{1}{3} \mid 4\right)$; bottom $\left(0, 0, \frac{1}{3}, \frac{4}{3} \mid 10\right)$.

$\max Z = 10$ at $(2, 4)$, so **$\min C = -10$ at (2, 4)**.

b. With $x = 0$: $y \le 6$ and $2y \le 6$, so $y = 3$, $C = -9$: worse than -10. Raising x loosens $-x + 2y \le 6$, letting y rise from 3 to 4; the extra y lowers C by 3, more than the 2 that x adds.

MARKING SCHEME

- ✓ $\min C = -\max(-C)$; bottom row 1, -3.
- ✓ y in, s_2 out; $Z = 9$.
- ✓ x in, s_1 out; $Z = 10$ at $(2, 4)$.
- ✓ $\min C = -10$.
- ✓ $x = 0$ gives only -9: x loosens the second constraint, so y can grow.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	11	9	**20**

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