Unit 15 · System of Linear Equations

HARDER PRACTICE QUESTIONS

Full marks 21

Write your answers on separate paper. Show your working. Write each row operation you use, such as $R_3 \to R_3 - 2R_1$.

1 Two unknown constants [5]

The system $x + y + z = 2$, $x + 2y + 3z = 5$, $2x + 3y + az = b$ has two constants, a and b.

a. Reduce it by Gauss elimination to upper triangular form.

b. For which values of a and b does it have exactly one solution, no solution, and infinitely many solutions?

c. In the last case, write every solution using $z = t$, and check one of them in all three equations.

2 A tiny pivot [6]

Solve $0.0003x + 1.566y = 1.569$, $0.3454x - 2.436y = 1.018$ by Gauss elimination, **rounding every result to four significant figures**, as a computer rounds to its own fixed number of digits.

a. Without swapping the equations: the multiplier is $m = \dfrac{0.3454}{0.0003}$. Find y, then x.

b. Now swap the equations first, so that the pivot is 0.3454, and solve again with the same rounding.

c. The exact solution is $x = 10$, $y = 1$. Which way is right, and why did the other go wrong, though its y was almost right?

3 A heated plate [6]

A square metal plate has its top edge held at 100 °C and its other three edges at 0 °C. Four points inside it, A and B in the top row and C and D below them (A above C, B above D), are equally spaced, so that each point's neighbors are two of the others and two edges. When the plate is steady, the temperature at each point is the mean of its four neighbors.

a. Show that $4A = 100 + B + C$, and write the other three equations.

b. Is the system diagonally dominant? Do two iterations of Gauss–Seidel from all zeros, in the order A, B, C, D, to three decimal places.

c. Use the plate's symmetry to find the exact temperatures.

d. A real plate is cut into 1000 points in each direction, a million unknowns. Elimination takes about $\frac{2}{3}n^3$ operations; one Gauss–Seidel iteration about 5 for each unknown. Compare the work for elimination with that for 1000 iterations.

4 When to stop [4]

Pratiksha solves $4x + y = 9$, $x + 3y = 5$ by Gauss–Seidel from $(0, 0)$ and stops when two iterations agree to two decimal places.

a. Work the iterations, and say where she stops and what she gives as the answer.

b. Her friend stops as soon as one iteration is within 0.01 of the one before in x, and gives x only. Find y from the second equation with that x, and say whether the pair fits the first equation to two decimal places.

c. Find the exact solution, and say what the agreement of two iterations does and does not promise.