Unit 15 · System of Linear Equations

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 21

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Two unknown constants

ANSWER HA · 5

Tests: reading the last row when it carries constants; that "infinitely many" needs **both** sides of the last row to be 0 (the tempting answer: $a = 4$ alone); writing an infinite family with a parameter.

a. $R_2 \to R_2 - R_1$: $y + 2z = 3$. $R_3 \to R_3 - 2R_1$: $y + (a-2)z = b - 4$. $R_3 \to R_3 - R_2$: $(a-4)z = b - 7$.

b. $a \neq 4$: $z = \dfrac{b-7}{a-4}$, then y and x: **exactly one** (for every b). $a = 4$, $b \neq 7$: the last row is $0 = b - 7 \neq 0$: **none**. $a = 4$ and $b = 7$: the last row is $0 = 0$: **infinitely many**.

c. $z = t$, $y = 3 - 2t$, $x = 2 - y - z = -1 + t$: $(x, y, z) = (-1 + t, 3 - 2t, t)$. With $t = 1$: $(0, 1, 1)$: $0 + 1 + 1 = 2$ ✓, $0 + 2 + 3 = 5$ ✓, $0 + 3 + 4 = 7$ ✓.

MARKING SCHEME

- ✓ Reduced to $(a-4)z = b - 7$.
- ✓ One solution for $a \neq 4$, any b.
- ✓ None for $a = 4$, $b \neq 7$; infinitely many for $a = 4$, $b = 7$.
- ✓ $(-1 + t, 3 - 2t, t)$.
- ✓ One solution checked in all three equations.

2 A tiny pivot

ANSWER HA · 6

Tests: partial pivoting; how a tiny pivot makes a huge multiplier that swamps the other numbers in rounding; why a small error in y becomes a large one in x (the tempting belief: rounding to four figures loses only the fourth figure).

a. $m = 1151$. $-2.436 - 1151 \times 1.566 = -2.436 - 1802 = -1804$;
$1.018 - 1151 \times 1.569 = 1.018 - 1806 = -1805$. $y = \dfrac{-1805}{-1804} = 1.001$.
$0.0003x = 1.569 - 1.566 \times 1.001 = 1.569 - 1.568 = 0.001$: $x = \mathbf{3.333}$.

b. Swapped: $m = \dfrac{0.0003}{0.3454} = 0.0008686$. $1.566 - 0.0008686 \times (-2.436) = 1.566 + 0.002116 = 1.568$;
$1.569 - 0.0008686 \times 1.018 = 1.569 - 0.0008842 = 1.568$. $y = 1.000$.
$0.3454x = 1.018 + 2.436 = 3.454$: $x = \mathbf{10.00}$.

c. b. is right. In a. the multiplier 1151 is so large that -2.436 and 1.018 vanish in the rounding beside 1802 and 1806: their information is lost. And x was found by dividing a small difference, $1.569 - 1.568$, by the tiny 0.0003, so y's error of 0.001 was multiplied by $\dfrac{1.566}{0.0003} \approx 5000$. With the large pivot, the multiplier is small and nothing is swamped. That is why a computer always pivots on the largest entry.

MARKING SCHEME

- ✓ $m = 1151$ and the second row -1804, -1805.
- ✓ $y = 1.001$, $x = 3.333$.
- ✓ Swapped: $m = 0.0008686$, the row 1.568, 1.568.
- ✓ $y = 1.000$, $x = 10.00$.
- ✓ The large multiplier swamped the other entries in the rounding.
- ✓ The small error in y was multiplied about 5000 times in x; pivot on the largest entry.

3 A heated plate

ANSWER

Tests: setting up a system from a rule ("mean of the neighbors"); checking dominance where the diagonal only just wins; symmetry as a check on iteration; why iteration wins for big, sparse systems.

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a. A's neighbors: the top edge (100), the left edge (0), B and C: $4A = 100 + B + C$. Likewise $4B = 100 + A + D$, $4C = A + D$, $4D = B + C$. A · 6

b. In each equation the diagonal coefficient is 4 and the others add to at most $1 + 1 = 2$: dominant. $A = \frac{100 + B + C}{4}$, $B = \frac{100 + A + D}{4}$, $C = \frac{A + D}{4}$, $D = \frac{B + C}{4}$.

Iteration 1: $A = 25$, $B = 31.25$, $C = 6.25$, $D = 9.375$.

Iteration 2: $A = 34.375$, $B = 35.938$, $C = 10.938$, $D = 11.719$.

c. The plate is the same from left to right, so $A = B$ and $C = D$. Then $4A = 100 + A + C$, $3A - C = 100$; $4C = A + C$, $A = 3C$. So $8C = 100$: $C = D = \mathbf{12.5\ °C}$, $A = B = \mathbf{37.5\ °C}$. The iterations are closing in on these from below (the next gives 36.7, 37.1, 12.1, 12.3).

d. Elimination: $\frac{2}{3}(10^6)^3 \approx 6.7 \times 10^{17}$ operations. Iteration: $1000 \times 5 \times 10^6 = 5 \times 10^9$. Elimination is about **a hundred million times** more work: each point links to only four others, and iteration uses just those, while elimination fills in the zeros.

MARKING SCHEME

- ✓ $4A = 100 + B + C$ explained from the neighbors.
- ✓ The other three equations.
- ✓ Dominant ($4 > 2$), with each unknown written from its equation.
- ✓ The two iterations.
- ✓ $A = B = 37.5$, $C = D = 12.5$ by symmetry.
- ✓ 6.7×10^{17} against 5×10^9: iteration wins on a sparse system.

4 When to stop

ANSWER

A · 4

Tests: a stopping rule applied to both unknowns and to the places asked; that agreement between iterations is a sign of closeness, not a proof.

a. $x = \frac{9 - y}{4}$, $y = \frac{5 - x}{3}$: (2.25, 0.917), (2.021, 0.993), (2.002, 0.999), (2.000, 1.000). To two places: (2.25, 0.92), (2.02, 0.99), (2.00, 1.00), (2.00, 1.00): the third and fourth agree. She stops at the fourth and gives $x = \mathbf{2.00}$, $y = \mathbf{1.00}$.

b. The x-values 2.021 and 2.002 differ by 0.019; 2.002 and 2.000 by 0.002, so he stops at $x = 2.000$. $y = \frac{5 - 2.000}{3} = 1.000$, and $4(2.000) + 1.000 = 9.000$ ✓: it fits.

c. Exact: $y = 9 - 4x$ in the second, $x + 27 - 12x = 5$, so $x = 2$, $y = 1$. Agreement of two iterations shows the values have stopped changing at that accuracy; for a dominant system, where each step shrinks the error, that means they are close to the solution. It is not a proof in general: checking in the equations is.

MARKING SCHEME

- ✓ The iterations, newest values used.
- ✓ Stops at the fourth: (2.00, 1.00).
- ✓ $y = 1.000$, checked in the first equation.
- ✓ Exact (2, 1); agreement shows the values settled, not a proof: check in the equations.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	10	11	**21**

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