Grade 12 Mathematics, Explained

Unit 15 · System of Linear Equations

ANSWER KEY · FOR TEACHERS

Full marks 24

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

1. upper triangular
2. back substitution
3. pivot
4. swapped
5. iterative
6. diagonally dominant

Each carries 0.5 mark. K · 3

2 True or false

i. ✓ **True** U · 3

ii. ✗ **False**: $0 = 0$ is always true: the system has infinitely many solutions (one equation fewer than unknowns); a row $(0\ 0\ 0 \mid c)$ with $c \neq 0$ means no solution.

iii. ✗ **False**: Gauss–Seidel always uses the **newest** values, those found earlier in the same iteration; using only the old ones is Jacobi's method.

iv. ✓ **True**

v. ✓ **True**

vi. ✗ **False**: Diagonal dominance is enough for convergence, but not necessary: some other systems converge too.

Each carries 0.5 mark.

3 Multiple choice

i. **b.** upper triangular K · 1

The entries below the pivots are cleared, leaving zeros under the diagonal. Clearing above as well, to a diagonal or the identity, is Gauss–Jordan's method.

ii. **b.** $4x - y = 2, x + 5y = 7$ U · 1

$|4| > |-1|$ in the first equation and $|5| > |1|$ in the second. The first option has the same equations in the order that puts the small coefficients on the diagonal.

iii. **a.** no solution U · 1

The row says $0x + 0y + 0z = 5$, that is $0 = 5$: impossible, so no solution.

iv. **b.** (2.25, 0.917) A · 1

$x = \frac{9 - 0}{4} = 2.25$, then with the new x: $y = \frac{5 - 2.25}{3} = 0.917$. (2.25, 1.667) uses the old $x = 0$ (Jacobi); (2, 1) is the exact solution, not one iteration.

v. **c.** $k = 4$ HA · 1

$R_2 - R_1$: $y + 2z = 3$; $R_3 - 2R_1$: $y + (k - 2)z = 3$; subtracting these, $(k - 4)z = 0$. For $k = 4$ the last row is $(0\ 0\ 0 \mid 0)$: infinitely many. For every other k, $z = 0$ and the solution is unique.

4 Short answers

a. MARKING SCHEME U · 2

✓ As written the system is not diagonally dominant.

✓ Swapped, each diagonal coefficient is the largest: dominant, so it converges.

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b. MARKING SCHEME A · 3

- ✓ Each equation solved for its own unknown.
- ✓ Iteration 1: 1.2, 1.08, 0.972 (newest values used).
- ✓ Iteration 2: 0.9948, 1.0033, 1.0002.

c. ANSWER A · 5

$$\left|\begin{array}{cccc} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & 2 & -1 & 2 \end{array}\right|$$

$R_2 \to R_2 - 2R_1$: $(0\ -3\ -1 \mid -9)$; $R_3 \to R_3 - R_1$: $(0\ 1\ -2 \mid -4)$.

$R_3 \to R_3 + \frac{1}{3}R_2$: $\left(0\ 0\ -\frac{7}{3} \mid -7\right)$.

Back substitution: $z = 3$; $-3y - 3 = -9$, $y = 2$; $x + 2 + 3 = 6$, $x = 1$.

$x = \mathbf{1}, y = \mathbf{2}, z = \mathbf{3}$. Check: $1 + 4 - 3 = 2$ ✓.

MARKING SCHEME

- ✓ The augmented matrix.
- ✓ $R_2 - 2R_1$ and $R_3 - R_1$.
- ✓ $R_3 + \frac{1}{3}R_2$, upper triangular.
- ✓ $z = 3$, $y = 2$ by back substitution.
- ✓ $x = 1$, checked.

5 Think

ANSWER HA · 3

$(-7, 42)$, $(161, -798)$, $(-3199, 16\,002)$: the values grow larger and larger, swinging from sign to sign, instead of settling at the solution $(1, 2)$.

Each step multiplies the error: x takes 4 times the error in y, and y 5 times the error in x, so every iteration makes the error about 20 times larger. With the equations swapped (diagonally dominant), each step divides it instead.

MARKING SCHEME

- ✓ The three iterations: $(-7, 42)$, $(161, -798)$, $(-3199, 16\,002)$.
- ✓ It diverges; the solution $(1, 2)$ is never approached.
- ✓ The large coefficients are off the diagonal, so each step multiplies the error (by about 20); swapping the equations fixes it.

6 Activity: Newest values or old ones?

(not marked)

WHAT STUDENTS SHOULD FIND

Gauss–Seidel: (2.25, 0.917), (2.021, 0.993), (2.002, 0.999), (2.000, 1.000) at the fourth or fifth iteration. Jacobi: (2.25, 1.667), (1.833, 0.917), (2.021, 1.056), (1.986, 0.993), (2.002, 1.005): it swings round the answer and needs about eight iterations. Using the newest values roughly halves the work.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	4	7	9	4	**24**

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