Unit 14 · Differential Equations

HARDER PRACTICE QUESTIONS

Full marks 24

Write your answers on separate paper. Show your working. In a solution, say what you divide by and when it may be 0.

1 How many constants? [4]

a. Shishir forms the differential equation of the family $y = Ae^{x+B}$. He sees two constants, differentiates twice and gets $\frac{d^2y}{dx^2} = \frac{dy}{dx}$. Show that his equation is satisfied by $y = 1 + e^x$, which is not in the family. Find the right equation, and explain his mistake.

b. Form the differential equation of all circles of radius 2 with their centers on the x-axis.

2 When was the tea made? [5]

Kritika finds a cup of tea on the table at 3:00 pm at 60 °C; at 3:10 pm it is at 45 °C. The room is at 20 °C, and the tea cools by Newton's law, $\frac{dT}{dt} = -k(T - 20)$. Tea is poured at about 95 °C.

a. Solve the equation with t counted in minutes from 3:00 pm, and find k.

b. At about what time was the tea poured?

c. Her brother says, "It lost 15 °C in 10 minutes, so to fall from 95 °C to 60 °C took about $35 \div 1.5 \approx 23$ minutes." Explain why his estimate is too long.

3 Salt in a tank [6]

A tank holds 500 liters of pure water. Salty water, with 0.1 kg of salt in each liter, runs in at 5 liters a minute; the tank is kept well stirred, and the mixture runs out at 5 liters a minute. Let S kg be the salt in the tank after t minutes.

a. Explain why $\frac{dS}{dt} = 0.5 - \frac{S}{100}$.

b. Solve it with $S = 0$ at $t = 0$.

c. When does the tank hold 25 kg of salt?

d. What happens to S after a long time? Explain why, without the solution, it could never go past that.

4 Make it exact [4]

a. Find the constant k for which $(2xy + ky^2)\,dx + (x^2 + 4xy)\,dy = 0$ is exact.

b. With that k, solve it, and find the solution through $(1, 1)$.

5 A curve from its tangents [5]

a. At every point $P(x, y)$ of a curve in the first quadrant, the piece of the tangent between P and the x-axis is bisected by the y-axis. Show that $\frac{dy}{dx} = \frac{y}{2x}$, and find the curve through $(2, 2)$.

b. At every point of another curve, the normal passes through the origin. Find the curves that do this.