Unit 14 · Differential Equations

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 24

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 How many constants?

ANSWER HA · 4

Tests: the order is the number of **independent** constants, not of letters; eliminating constants by differentiating.

a. $y = 1 + e^x$: $y'' - y' = e^x - e^x = 0$ ✓, but $1 + e^x$ is not Ae^{x+B} for any A, B: every member of the family is a multiple of e^x, and $1 + e^x$ is not. His equation has too many solutions.

$Ae^{x+B} = (Ae^B)e^x$: A and B only ever appear as one constant, $K = Ae^B$. One constant, so order 1: $y' = Ke^x = y$, that is $\dfrac{dy}{dx} = y$.

b. $(x - a)^2 + y^2 = 4$, one constant a (the radius is fixed). Differentiate: $(x - a) + yy' = 0$, so $x - a = -yy'$. Put back: $y^2(y')^2 + y^2 = 4$, that is $\mathbf{y^2\left[1 + \left(\dfrac{dy}{dx}\right)^2\right] = 4}$.

MARKING SCHEME

- ✓ $y = 1 + e^x$ satisfies $y'' = y'$ but is not in the family.
- ✓ A and B combine into one constant Ae^B, so the order is 1: $y' = y$.
- ✓ $(x - a)^2 + y^2 = 4$ differentiated: $x - a = -yy'$.
- ✓ $y^2\left[1 + (y')^2\right] = 4$.

2 When was the tea made?

ANSWER HA · 5

Tests: Newton's cooling with the clock started at an unknown moment; a time before the first reading (negative t); why the cooling is not at a steady rate.

a. $\dfrac{dT}{T - 20} = -k\,dt$, $T = 20 + 40e^{-kt}$ (60 °C at $t = 0$). At $t = 10$: $25 = 40e^{-10k}$, $k = \dfrac{1}{10}\ln\dfrac{8}{5} = \mathbf{0.0470\ a\ minute}$.

b. $95 = 20 + 40e^{-kt}$: $e^{-kt} = \dfrac{75}{40}$, $t = -\dfrac{\ln 1.875}{k} = -\dfrac{10\ln 1.875}{\ln 1.6} \approx -13.4$ minutes. Poured about 13 minutes before 3:00 pm: **about 2:47 pm**.

c. The rate of cooling is $k(T - 20)$: proportional to how much hotter than the room the tea is. Between 95 °C and 60 °C the tea was much hotter than between 60 °C and 45 °C, so it cooled faster, and the 35 °C took only about 13 minutes, not 23. (The tempting answer: cooling at a steady 1.5 °C a minute.)

MARKING SCHEME

- ✓ $T = 20 + 40e^{-kt}$.
- ✓ $k = \dfrac{1}{10}\ln 1.6 \approx 0.047$.
- ✓ $e^{-kt} = 1.875$, $t \approx -13.4$ minutes.
- ✓ About 2:47 pm.
- ✓ The hotter tea cooled faster (rate proportional to $T - 20$), so less than 23 minutes.

3 Salt in a tank

ANSWER

Tests: writing a differential equation from a description (rate in minus rate out); a linear (here also separable) equation with a condition; the long-run value and its meaning.

a. Salt comes in at $5 \times 0.1 = 0.5$ kg a minute. The tank always holds 500 L, so each liter going out carries $\dfrac{S}{500}$ kg: out at $5 \times \dfrac{S}{500} = \dfrac{S}{100}$ kg a minute. The change is in minus out.

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b. $\frac{dS}{dt} + \frac{S}{100} = 0.5$: IF $= e^{t/100}$, $Se^{t/100} = \int 0.5e^{t/100}\,dt = 50e^{t/100} + C$. $S(0) = 0$: $C = -50$. **A · 6**
$S = \mathbf{50}\left(\mathbf{1} - e^{-t/\mathbf{100}}\right)$.

c. $25 = 50(1 - e^{-t/100})$: $e^{-t/100} = \frac{1}{2}$, $t = 100 \ln 2 \approx$ **69.3 minutes**.

d. $e^{-t/100} \to 0$, so $S \to$ **50 kg**: 0.1 kg/L in all 500 L, the concentration of what flows in. If S reached 50 kg, $\frac{dS}{dt} = 0$: salt would leave as fast as it came; above 50 kg, $\frac{dS}{dt} < 0$. Stirring in water of 0.1 kg/L can't make the tank saltier than 0.1 kg/L.

MARKING SCHEME

- ✓ In: 0.5 kg a minute; out: $5 \times \frac{S}{500} = \frac{S}{100}$.
- ✓ IF $e^{t/100}$ (or separated).
- ✓ $S = 50(1 - e^{-t/100})$.
- ✓ $t = 100 \ln 2 \approx 69.3$ minutes.
- ✓ $S \to 50$ kg.
- ✓ At 50 kg the rate is 0, above it negative: the tank can't be saltier than what flows in.

4 Make it exact

ANSWER **A · 4**

Tests: the test for exactness as an equation for a constant; the unit's rule for an exact equation without counting a term twice.

a. $\frac{\partial M}{\partial y} = 2x + 2ky$, $\frac{\partial N}{\partial x} = 2x + 4y$. Equal for every x, y only when $k = 2$.

b. $\int M\,dx$ (y held constant) $= x^2y + 2xy^2$. Every term of $N = x^2 + 4xy$ has x in it, so nothing more is added (the x^2 and $4xy$ are already the y-derivatives of $x^2y + 2xy^2$).

General solution: $\mathbf{x^2y + 2xy^2 = C}$. Through (1, 1): $C = 3$, so $x^2y + 2xy^2 = 3$.

(Check: $\frac{\partial}{\partial x}(x^2y + 2xy^2) = 2xy + 2y^2 = M$, $\frac{\partial}{\partial y} = x^2 + 4xy = N$. ✓)

MARKING SCHEME

- ✓ $\frac{\partial M}{\partial y} = 2x + 2ky$ and $\frac{\partial N}{\partial x} = 2x + 4y$.
- ✓ $k = 2$.
- ✓ $x^2y + 2xy^2 = C$, no term counted twice.
- ✓ $C = 3$.

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5 A curve from its tangents

ANSWER HA · 5

Tests: turning a geometric condition on the tangent or normal into a differential equation; solving it.

a. The tangent at P meets the x-axis where $0 - y = y'(X - x)$, at $X = x - \frac{y}{y'}$. The midpoint of $P(x, y)$ and $(X, 0)$ is on the y-axis when $x + X = 0$: $2x - \frac{y}{y'} = 0$, so $\frac{dy}{dx} = \frac{y}{2x}$ ✓.

$\frac{dy}{y} = \frac{dx}{2x}$: $\ln y = \frac{1}{2}\ln x + c$, $y^2 = Cx$. Through $(2, 2)$: $C = 2$. $\mathbf{y^2 = 2x}$, a parabola.

b. The normal at (x, y) has slope $-\frac{1}{y'}$; through the origin it has slope $\frac{y}{x}$. So $-\frac{1}{y'} = \frac{y}{x}$, $y\frac{dy}{dx} = -x$, and $y\,dy = -x\,dx$: $x^2 + y^2 = C$. **Circles with center at the origin**, as expected: every radius of a circle is a normal.

MARKING SCHEME

✓ The tangent meets the x-axis at $x - \frac{y}{y'}$.

✓ The midpoint condition gives $y' = \frac{y}{2x}$.

✓ $y^2 = 2x$ through $(2, 2)$.

✓ The normal through O: $yy' = -x$.

✓ $x^2 + y^2 = C$, circles about the origin.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	10	14	**24**

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