Unit 14 · Differential Equations

ANSWER KEY · FOR TEACHERS

Full marks 35

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

K · 4

1. order
2. degree
3. arbitrary
4. general
5. particular
6. separable
7. homogeneous
8. integrating factor

Each carries 0.5 mark.

2 True or false

U · 3

i. ✓ **True**

ii. ✗ **False**: Squared to clear the root, $\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$: the degree is 2.

iii. ✓ **True**

iv. ✗ **False**: Divide by x first: $P = -\frac{2}{x}$, so the IF is $e^{-2\ln x} = \frac{1}{x^2}$ $(x > 0)$.

v. ✓ **True**

vi. ✗ **False**: Many are not, such as $\frac{dy}{dx} = x + y$; they need another method (here, the linear one).

Each carries 0.5 mark.

3 Multiple choice

i. **b.** 2 and 2 — K · 1

The highest derivative is the second (order 2), and it is squared (degree 2).

ii. **b.** x^2 — U · 1

$e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$. $2\ln x$ is $\int P\,dx$, before taking e to that power.

iii. **a.** $y = Cx$ — U · 1

$\frac{dy}{y} = \frac{dx}{x}$, $\ln|y| = \ln|x| + c$, $y = Cx$: every line through the origin. $xy = C$ solves $\frac{dy}{dx} = -\frac{y}{x}$.

iv. **b.** $e^y = e^x + C$ — A · 1

$e^y\,dy = e^x\,dx$, so $e^y = e^x + C$. $y = x + C$ would mean $e^y = e^C\,e^x$, a multiple of e^x rather than e^x plus a constant; it fits only when $C = 0$.

v. **a.** $y = -x - 1$ — HA · 1

With $C = 0$: $y = -x - 1$. Check: $\frac{dy}{dx} = -1$ and $x + y = -1$. ✓ The last option forgets that C may be 0.

4 Short answers

a. MARKING SCHEME — A · 2

✓ Differentiated twice.

✓ $y'' + y = 0$.

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b. MARKING SCHEME U · 2

✓ $y = 0$ checked in the equation.

✓ Dividing by y^2 assumed $y \neq 0$; no C gives $y = 0$.

c. MARKING SCHEME A · 3

✓ Separated and integrated: $\tan^{-1} y = \tan^{-1} x + C$.

✓ $C = \frac{\pi}{4}$.

✓ $\tan^{-1} y = \tan^{-1} x + \frac{\pi}{4}$, or $y = \frac{1+x}{1-x}$.

d. ANSWER A · 5

Standard form: $\frac{dy}{dx} + \frac{2}{x}y = x$. IF $= e^{\int \frac{2}{x}dx} = x^2$.

$yx^2 = \int x \cdot x^2\,dx = \frac{x^4}{4} + C$, so $y = \frac{x^2}{4} + \frac{C}{x^2}$.

Through $(1, 1)$: $1 = \frac{1}{4} + C$, $C = \frac{3}{4}$: $y = \mathbf{\frac{x^2}{4} + \frac{3}{4x^2}}$.

MARKING SCHEME

✓ Standard form with $P = \frac{2}{x}$, $Q = x$.

✓ IF $= x^2$.

✓ $yx^2 = \frac{x^4}{4} + C$.

✓ General solution $y = \frac{x^2}{4} + \frac{C}{x^2}$.

✓ $C = \frac{3}{4}$.

5 Long answer

ANSWER A · 8

a. Homogeneous: put $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$. For $x > 0$, $\sqrt{x^2 + v^2x^2} = x\sqrt{1+v^2}$:

$x\left(v + x\frac{dv}{dx}\right) = vx + x\sqrt{1+v^2}$, so $x\frac{dv}{dx} = \sqrt{1+v^2}$ and $\frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$.

$\ln\left(v + \sqrt{1+v^2}\right) = \ln x + \ln C$, so $v + \sqrt{1+v^2} = Cx$.

With $v = \frac{y}{x}$: $\frac{y + \sqrt{x^2+y^2}}{x} = Cx$, that is $\mathbf{y + \sqrt{x^2+y^2} = Cx^2}$.

b. $\frac{dT}{T-20} = -k\,dt$: $\ln(T-20) = -kt + c$, so $T = 20 + 70e^{-kt}$ ($T = 90$ at $t = 0$).

At $t = 10$: $40 = 70e^{-10k}$, $e^{-10k} = \frac{4}{7}$, $k = \frac{1}{10}\ln\frac{7}{4} = 0.0560$ a minute.

At 40 °C: $20 = 70e^{-kt}$, $e^{-kt} = \frac{2}{7}$, $t = \frac{\ln(7/2)}{k} = \frac{10\ln 3.5}{\ln 1.75} \approx$ **22.4 minutes**.

What it means: the first 30 °C of cooling took 10 minutes, the next 20 °C another 12.4: the closer the tea gets to the room, the slower it cools.

MARKING SCHEME

✓ $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$, with $\sqrt{x^2(1+v^2)} = x\sqrt{1+v^2}$ for $x > 0$.

✓ Separated: $\frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$.

✓ $\ln\left(v + \sqrt{1+v^2}\right) = \ln Cx$.

✓ $y + \sqrt{x^2+y^2} = Cx^2$.

✓ $T = 20 + 70e^{-kt}$.

✓ $e^{-10k} = \frac{4}{7}$, $k \approx 0.056$.

✓ $e^{-kt} = \frac{2}{7}$.

✓ About 22.4 minutes.

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6 Think

ANSWER HA · 3

$y = 0$: $\frac{dy}{dx} = 0 = 2\sqrt{0}$ ✓. Through (0, 0): $y = 0$, and $y = x^2$ ($C = 0$, for $x \geq 0$): $\frac{dy}{dx} = 2x = 2\sqrt{x^2}$ ✓. Two solutions through one point.

Dividing by $\sqrt{y}$ assumed $y \neq 0$: it lost the solution $y = 0$, which no C gives back. Check what you divide by, and where the equation is not smooth ($\sqrt{y}$ has no derivative at 0), solutions may meet.

MARKING SCHEME

- ✓ $y = 0$ checked.
- ✓ $y = x^2$ and $y = 0$ both through (0, 0).
- ✓ Dividing by $\sqrt{y}$ lost $y = 0$.

7 Activity: A cooling cup

(not marked)

WHAT STUDENTS SHOULD FIND

By Newton's law, $\ln(T - T_s) = \ln(T_0 - T_s) - kt$: the points lie close to a falling straight line, whose slope is $-k$ (for a cup of water, roughly 0.02 to 0.05 a minute, depending on the cup and whether it has a lid). The early points may fall a little faster than the line, while the cup itself warms up and steam carries heat away.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	5	7	19	4	**35**

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