Unit 13 · Anti-derivatives

HARDER PRACTICE QUESTIONS

Full marks 23

Write your answers on separate paper. Show your working. Check an anti-derivative by differentiating it where you can.

1 Once round the circle [5]

For $a > b > 0$, $\int \frac{dx}{a + b\cos x} = F(x) + C$, where $F(x) = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan\frac{x}{2}\right)$.

a. Nischal works out $\int_0^{2\pi} \frac{dx}{a + b\cos x} = F(2\pi) - F(0)$. What does he get, and why can't it be right?

b. Show that $\int_0^{2\pi} \frac{dx}{a + b\cos x} = \frac{2\pi}{\sqrt{a^2 - b^2}}$.

c. Use b. to find $\int_0^{2\pi} \frac{dx}{5 + 4\cos x}$.

2 The top in two parts [4]

Integrate $\int \frac{2\sin x + 3\cos x}{\sin x + \cos x}\,dx$. Then check your answer by differentiating it.

3 A fifth power below [4]

a. Integrate $\int \frac{dx}{x(x^5 + 1)}$, for $x > 0$, by first multiplying the top and the bottom by x^4.

b. Abinash tries partial fractions on $\frac{1}{x(x^5 + 1)}$ as it stands. Explain why the substitution is the better route.

4 Four answers to one integral [4]

Four students give $\int \frac{dx}{\sqrt{x^2 + 4}}$ as:

Sanskriti: $\ln\left(x + \sqrt{x^2 + 4}\right) + C$; Rehan: $\sinh^{-1}\frac{x}{2} + C$; Nima: $-\ln\left(\sqrt{x^2 + 4} - x\right) + C$; Yugin: $\ln\left|x + \sqrt{x^2 - 4}\right| + C$.

Which of them are right? For each right answer, find by how much it differs from Sanskriti's.

5 The dip chart again [6]

A diesel tank is a cylinder lying on its side, its round end of radius 1 m, 5 m long (15,708 liters when full). A dip rod reading h meters wets the part of the end below $y = h - 1$, measured from the center, so the wet area of the end is $A(h) = \int_{-1}^{h-1} 2\sqrt{1 - y^2}\,dy$.

a. Using $\int \sqrt{a^2 - y^2}\,dy = \frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}\frac{y}{a} + C$, find $A(1)$, and say why the answer is what you would expect.

b. Show that $A(h) + A(2 - h) = \pi$, and explain what it means in the tank.

c. A dip of 0.5 m shows 3,071 liters. Without integrating, find the liters at a dip of 1.5 m.

d. Find the liters at a dip of 1.2 m, and the share of the full tank.