Unit 13 · Anti-derivatives

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 23

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Once round the circle

ANSWER HA · 5

Tests: an answer found with $t = \tan\frac{x}{2}$ holds only between odd multiples of π; a definite integral across π must be split, with a limit at π; symmetry of the integrand.

a. $\tan\pi = 0$ and $\tan 0 = 0$, so $F(2\pi) - F(0) = 0$. But $a + b\cos x \geq a - b > 0$, so the integrand is positive everywhere and the integral must be positive. F is not one anti-derivative over $[0, 2\pi]$: $\tan\frac{x}{2}$ is not defined at $x = \pi$, where F jumps.

b. $\cos(2\pi - x) = \cos x$, so the integrand takes the same values on $[\pi, 2\pi]$ as on $[0, \pi]$: the integral is $2\int_0^{\pi}$. On $[0, \pi)$, F holds; as $x \to \pi^-$, $\tan\frac{x}{2} \to +\infty$ and $\tan^{-1} \to \frac{\pi}{2}$, so

$F \to \frac{2}{\sqrt{a^2 - b^2}} \cdot \frac{\pi}{2} = \frac{\pi}{\sqrt{a^2 - b^2}}$, and $F(0) = 0$.

$\int_0^{2\pi} = 2 \times \frac{\pi}{\sqrt{a^2 - b^2}} = \frac{2\pi}{\sqrt{a^2 - b^2}}$ ✓

c. $\frac{2\pi}{\sqrt{25 - 16}} = \frac{\mathbf{2\pi}}{\mathbf{3}}$ (about 2.09).

MARKING SCHEME

✓ Nischal gets 0.

✓ A positive integrand can't give 0; F jumps at π, where $\tan\frac{x}{2}$ is undefined.

✓ The integral is $2\int_0^{\pi}$ by symmetry.

✓ $F \to \frac{\pi}{\sqrt{a^2 - b^2}}$ as $x \to \pi^-$, giving $\frac{2\pi}{\sqrt{a^2 - b^2}}$.

✓ $\frac{2\pi}{3}$.

2 The top in two parts

ANSWER A · 4

Tests: writing the top as $A \times (\text{bottom}) + B \times (\text{bottom})'$, so that one part is a constant and the other a log.

The bottom's derivative is $\cos x - \sin x$. Let $2\sin x + 3\cos x = A(\sin x + \cos x) + B(\cos x - \sin x)$.

$\sin x$: $A - B = 2$; $\cos x$: $A + B = 3$. So $A = \frac{5}{2}$, $B = \frac{1}{2}$.

$\int = \frac{5}{2}\int dx + \frac{1}{2}\int\frac{\cos x - \sin x}{\sin x + \cos x}\,dx = \mathbf{\frac{5}{2}x + \frac{1}{2}\ln|\sin x + \cos x|} + C$.

Check: $\frac{5}{2} + \frac{1}{2} \cdot \frac{\cos x - \sin x}{\sin x + \cos x} = \frac{5\sin x + 5\cos x + \cos x - \sin x}{2(\sin x + \cos x)} = \frac{4\sin x + 6\cos x}{2(\sin x + \cos x)}$ ✓

MARKING SCHEME

✓ The top as $A(\sin x + \cos x) + B(\cos x - \sin x)$.

✓ $A = \frac{5}{2}$, $B = \frac{1}{2}$.

✓ $\frac{5}{2}x + \frac{1}{2}\ln|\sin x + \cos x| + C$.

✓ Checked by differentiating.

3 A fifth power below

ANSWER

Tests: a substitution that makes a rational function, and seeing the hidden derivative ($x^4\,dx$); choosing a route.

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a. $\int \frac{x^4\,dx}{x^5(x^5+1)}$. Let $t = x^5$, $dt = 5x^4\,dx$: $\frac{1}{5}\int \frac{dt}{t(t+1)} = \frac{1}{5}\int\left(\frac{1}{t} - \frac{1}{t+1}\right)dt = \frac{1}{5}\ln\frac{t}{t+1} + C$. **HA · 4**

So $\int = \mathbf{\frac{1}{5}\ln\frac{x^5}{x^5+1}} + C$. (Check: $\frac{1}{5}\left(\frac{5}{x} - \frac{5x^4}{x^5+1}\right) = \frac{x^5+1-x^5}{x(x^5+1)}$ ✓)

b. $x^5 + 1$ factors as $(x+1)(x^4 - x^3 + x^2 - x + 1)$, and the quartic as two quadratics with irrational coefficients: five constants to find and several $\tan^{-1}$ and log pieces. The substitution sees that x^5 is all that matters and leaves a two-term split.

MARKING SCHEME

✓ $t = x^5$, $dt = 5x^4\,dx$, giving $\frac{1}{5}\int\frac{dt}{t(t+1)}$.

✓ $\frac{1}{t} - \frac{1}{t+1}$.

✓ $\frac{1}{5}\ln\frac{x^5}{x^5+1} + C$.

✓ Direct partial fractions need the factors of $x^5 + 1$ (a quartic with no simple factors): far longer.

4 Four answers to one integral

ANSWER **HA · 4**

Tests: anti-derivatives are a family, so answers that look different can differ by a constant; the log form of $\sinh^{-1}$; rationalising.

Sanskriti's is form 6 with $a = 2$: right.

Rehan: $\sinh^{-1}\frac{x}{2} = \ln\left(\frac{x}{2} + \sqrt{\frac{x^2}{4}+1}\right) = \ln\frac{x+\sqrt{x^2+4}}{2}$, which is Sanskriti's **minus ln 2**: right.

Nima: $\frac{1}{\sqrt{x^2+4}-x} = \frac{\sqrt{x^2+4}+x}{(x^2+4)-x^2} = \frac{x+\sqrt{x^2+4}}{4}$, so $-\ln\left(\sqrt{x^2+4}-x\right) = \ln\frac{x+\sqrt{x^2+4}}{4}$: Sanskriti's **minus ln 4**: right.

Yugin's is the form for $\frac{1}{\sqrt{x^2-4}}$: its derivative is $\frac{1}{\sqrt{x^2-4}}$, not $\frac{1}{\sqrt{x^2+4}}$ (it isn't even defined for $|x| < 2$): **wrong**.

MARKING SCHEME

✓ Sanskriti's is right (form 6).

✓ Rehan's is right, ln 2 less.

✓ Nima's is right, ln 4 less, by rationalising.

✓ Yugin's is wrong (it is $\int\frac{dx}{\sqrt{x^2-4}}$).

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5 The dip chart again

ANSWER A · 6

Tests: a definite integral as an area with a meaning; form 8 at numerical limits; using symmetry instead of computing.

With $G(y) = y\sqrt{1-y^2} + \sin^{-1} y$ (form 8 times 2, $a = 1$): $A(h) = G(h-1) - G(-1) = G(h-1) + \frac{\pi}{2}$.

a. $A(1) = G(0) + \frac{\pi}{2} = \frac{\pi}{\mathbf{2}}$: half of the circle's area π, as it must be for a rod reading the radius (half full).

b. G is odd: $G(-y) = -G(y)$. $A(2-h) = G(1-h) + \frac{\pi}{2} = -G(h-1) + \frac{\pi}{2}$. Adding: $A(h) + A(2-h) = \pi$ ✓. In the tank: the fuel at a dip h and the **empty** space at a dip $2-h$ are mirror images, so they fill the tank together.

c. $15\,708 - 3071 =$ **12,637 liters**.

d. $y = 0.2$: $G(0.2) = 0.2 \times 0.9798 + \sin^{-1} 0.2 = 0.1960 + 0.2014 = 0.3973$, $A = 0.3973 + 1.5708 = 1.9681$ m^2, volume $1.9681 \times 5 = 9.84$ m^3: **about 9,841 liters**, $\frac{1.9681}{\pi} =$ **62.6%** of the tank, though the rod reads 60% of the depth.

MARKING SCHEME

✓ $A(h) = G(h-1) + \frac{\pi}{2}$ from form 8.

✓ $A(1) = \frac{\pi}{2}$, half the circle.

✓ G odd, so $A(h) + A(2-h) = \pi$.

✓ The fuel at h and the empty part at $2-h$ are mirror images.

✓ 12,637 liters at 1.5 m.

✓ About 9,841 liters, 62.6%, at 1.2 m.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	10	13	**23**

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