Unit 13 · Anti-derivatives

ANSWER KEY · FOR TEACHERS

Full marks 34

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

K · 3

1. $\int \frac{dx}{x^2+a^2} \to$ **d.** $\frac{1}{a}\tan^{-1}\frac{x}{a}+C$
2. $\int \frac{dx}{\sqrt{a^2-x^2}} \to$ **e.** $\sin^{-1}\frac{x}{a}+C$
3. $\int \frac{dx}{x^2-a^2} \to$ **f.** $\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$
4. $\int \frac{dx}{\sqrt{x^2+a^2}} \to$ **b.** $\ln\left|x+\sqrt{x^2+a^2}\right|+C$
5. $\int \frac{dx}{x\sqrt{x^2-a^2}}, x>a \to$ **g.** $\frac{1}{a}\sec^{-1}\frac{x}{a}+C$
6. $\int \frac{dx}{a+b\cos x} \to$ **a.** the substitution $t=\tan\frac{x}{2}$

Each carries 0.5 mark.

2 True or false

U · 3

i. ✓ **True**

ii. ✗ **False**: It is improper (the top's degree is not less than the bottom's): divide first, $x+\frac{x}{x^2-1}$, then split the proper part.

iii. ✓ **True**

iv. ✗ **False**: $\frac{1}{2}\tan^{-1}\frac{x}{2}+C$: the $\frac{1}{a}$ in front was lost.

v. ✓ **True**

vi. ✗ **False**: $\tan\frac{x}{2}$ is not defined at $x=\pi$: the range must be split there.

Each carries 0.5 mark.

3 Multiple choice

i. K · 1

b. $\frac{1}{3}\tan^{-1}\frac{x}{3}+C$

Form 1 with $a=3$. The log is for x^2-9, a difference.

ii. U · 1

a. $\frac{1}{2}\tan^{-1}\frac{x-3}{2}+C$

$x^2-6x+13=(x-3)^2+4$, so $a=2$ and $u=x-3$. The log would need real roots, which there are none.

iii. U · 1

c. $\frac{1}{6}\tan^{-1}\frac{2x}{3}+C$

$\frac{1}{4}\int\frac{dx}{x^2+\frac{9}{4}}=\frac{1}{4}\cdot\frac{2}{3}\tan^{-1}\frac{2x}{3}=\frac{1}{6}\tan^{-1}\frac{2x}{3}$. The first forgets to take the 4 out.

iv. A · 1

a. $\ln\left|\frac{x}{x+1}\right|+C$

$\frac{1}{x(x+1)}=\frac{1}{x}-\frac{1}{x+1}$, so $\ln|x|-\ln|x+1|$. The third has the sign reversed.

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v. **b.** $\frac{1}{2}\ln\left|\frac{x-3}{x-1}\right| + C$ A·1

$(x-2)^2 - 1$, form 3 with $u = x-2$, $a = 1$: $\frac{1}{2}\ln\left|\frac{x-2-1}{x-2+1}\right|$. Completing the square to $(x-2)^2 + 1$ by mistake gives the $\tan^{-1}$; the first is upside down.

4 Short answers

a. MARKING SCHEME U·2

- ✓ The fraction is improper (degree 3 over degree 2); partial fractions need a proper one.
- ✓ $x + \frac{x}{x^2-1}$.

b. MARKING SCHEME A·2

- ✓ $(x+1)^2 + 4$.
- ✓ $\frac{1}{2}\tan^{-1}\frac{x+1}{2} + C$.

c. MARKING SCHEME A·3

- ✓ Her answer differentiates to $\frac{1}{x^2+9}$.
- ✓ The 4 of $4x^2$ was ignored; it must be taken out ($a = \frac{3}{2}$).
- ✓ $\frac{1}{6}\tan^{-1}\frac{2x}{3} + C$.

d. ANSWER A·5

The bottom's derivative is $2x + 2$. Write $3x + 5 = \frac{3}{2}(2x+2) + 2$.

$\int\frac{3x+5}{x^2+2x+5}\,dx = \frac{3}{2}\int\frac{2x+2}{x^2+2x+5}\,dx + 2\int\frac{dx}{(x+1)^2+4}$

$= \frac{3}{2}\ln(x^2+2x+5) + 2\cdot\frac{1}{2}\tan^{-1}\frac{x+1}{2} + C =$ \dfrac{3}{2}\mathrm{ln}(x^2 + 2x + 5) + \tan^{-1}\dfrac{x + 1}{2} + C.

(No modulus is needed: $x^2 + 2x + 5 > 0$ for every x.)

MARKING SCHEME

- ✓ $3x + 5 = \frac{3}{2}(2x+2) + 2$.
- ✓ Split into a log part and a standard part.
- ✓ $\frac{3}{2}\ln(x^2+2x+5)$.
- ✓ $(x+1)^2 + 4$ completed.
- ✓ $\tan^{-1}\frac{x+1}{2}$, with $+ C$.

5 Long answer

ANSWER

a. $\frac{x+5}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{D}{(x+2)^2}$, so $x + 5 = A(x+2)^2 + B(x-1)(x+2) + D(x-1)$.

$x = 1$: $6 = 9A$, $A = \frac{2}{3}$. $x = -2$: $3 = -3D$, $D = -1$. The x^2 terms: $0 = A + B$, $B = -\frac{2}{3}$.

$\int = \frac{2}{3}\ln|x-1| - \frac{2}{3}\ln|x+2| + \frac{1}{x+2} + C =$ \frac{2}{3}\mathrm{ln}\left|\frac{x - 1}{x + 2}\right| + \frac{1}{x + 2} + C.

b. $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2\,dt}{1+t^2}$: $5 + 4\cos x = \frac{5+5t^2+4-4t^2}{1+t^2} = \frac{9+t^2}{1+t^2}$.

$\int\frac{2\,dt}{9+t^2} = \frac{2}{3}\tan^{-1}\frac{t}{3} + C =$ \dfrac{2}{3}\tan^{-1}\left(\dfrac{1}{3}\tan\dfrac{x}{2}\right) + C.

c. From 0 to $\frac{\pi}{2}$, $\tan\frac{x}{2}$ goes from 0 to 1 (no odd multiple of π in between):

$\frac{2}{3}\tan^{-1}\frac{1}{3} - 0 = \frac{2}{3}\times 0.3218 \approx$ **0.2145**.

MARKING SCHEME

- ✓ The three partial fractions set up.

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✓ $A = \frac{2}{3}, D = -1$. A · 8

✓ $B = -\frac{2}{3}$.

✓ The integral, with $\int \frac{-dx}{(x+2)^2} = \frac{1}{x+2}$.

✓ $\cos x$ and dx in t.

✓ The integrand becomes $\frac{2}{9+t^2}$.

✓ $\frac{2}{3} \tan^{-1}\left(\frac{1}{3} \tan \frac{x}{2}\right) + C$.

✓ $\frac{2}{3} \tan^{-1} \frac{1}{3} \approx 0.2145$.

6 Think

ANSWER HA · 3

$\frac{1}{x^2}$ is positive everywhere it is defined, so an integral of it over a range must be positive, not -2.

$F(b) - F(a)$ needs an anti-derivative that holds over the whole range, but $-\frac{1}{x}$ (and $\frac{1}{x^2}$ itself) is not defined at $x = 0$, inside $[-1, 1]$. As with $t = \tan \frac{x}{2}$ at π, the range would have to be split at 0, and there the integrand grows without bound: the area is not a finite number at all.

MARKING SCHEME

✓ A positive integrand can't give a negative integral.

✓ $-\frac{1}{x}$ and $\frac{1}{x^2}$ are not defined at 0, inside the range.

✓ The range must be split at 0, where the integrand grows without bound (no finite area).

7 Activity: A dip chart for a bottle

(not marked)

WHAT STUDENTS SHOULD FIND

The share of the volume grows slowly at first, then faster, then slowly again: the round end is narrow near the bottom and widest in the middle. A quarter of the depth holds only about a fifth of the water (19.6% for a perfect circle), half the depth exactly half, three quarters about 80%. A bottle with a shaped bottom or shoulders gives a little less exact numbers.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	4	7	20	3	**34**

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