Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Two constants to choose

ANSWER HA · 4

Tests: checking the form before every use of the rule; a limit stays finite only if each stage is $\frac{0}{0}$, which fixes the constants.

At $x = 0$ the top is $a + b - 2$. Unless $a + b = 2$, the quotient is about $\frac{a+b-2}{x^4}$, without bound. So $a + b = 2$, and the form is $\frac{0}{0}$.

Once: $\frac{a\sinh x - b\sin x}{4x^3}$, $\frac{0}{0}$ at 0 whatever a and b.

Twice: $\frac{a\cosh x - b\cos x}{12x^2}$. The top at 0 is $a - b$; unless $a = b$ this grows without bound. So $a = b$, and with $a + b = 2$: $a = b = \mathbf{1}$; the form is $\frac{0}{0}$.

Three times: $\frac{\sinh x + \sin x}{24x}$, $\frac{0}{0}$.

Four times: $\frac{\cosh x + \cos x}{24} \to \frac{2}{24} = \mathbf{\frac{1}{12}}$.

MARKING SCHEME

- ✓ $a + b = 2$ for the first $\frac{0}{0}$.
- ✓ After two uses, $a - b = 0$ is needed: $a = b = 1$.
- ✓ Each later step checked to be $\frac{0}{0}$.
- ✓ The limit $\frac{1}{12}$.

2 Tape or calipers?

ANSWER HA · 5

Tests: for $y = kx^n$, the percentage error is n times that of x; what decides is the **relative** error of what is measured, not its absolute error.

a. $C = 2\pi r$, so $V = \frac{4}{3}\pi r^3 = \frac{C^3}{6\pi^2}$: $dV = \frac{3C^2}{6\pi^2}\,dC$, and $\frac{dV}{V} = 3\,\frac{dC}{C} = 3 \times \frac{0.5}{70} = 0.0214$: **2.1%**.

b. $d = \frac{70}{\pi} = 22.3$ cm. $V = \frac{\pi d^3}{6}$, so $\frac{dV}{V} = 3\,\frac{dd}{d} = 3 \times \frac{0.2}{22.3} = 0.0269$: **2.7%**.

c. No. The volume's percentage error is three times the **percentage** error of what is measured. The circumference is π times longer than the diameter, so its 0.5 cm is only 0.71% of it, while the calipers' 0.2 cm is 0.90% of the diameter. The tape gives the better volume.

MARKING SCHEME

- ✓ $\frac{dV}{V} = 3\frac{dC}{C}$.
- ✓ 2.1% from the circumference.
- ✓ $d = 22.3$ cm and $\frac{dV}{V} = 3\frac{dd}{d}$.
- ✓ 2.7% from the diameter.
- ✓ No: the relative error decides (0.71% against 0.90%); the tape is better.

3 Lines through a point off the curve

ANSWER

Tests: a tangent or normal through a given point not on the curve, with the point of contact unknown; not dividing by an unknown that may be 0.

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a. At (p, p^2) the tangent is $y - p^2 = 2p(x - p)$, that is $y = 2px - p^2$. Through $(1, -3)$: $-3 = 2p - p^2$, $p^2 - 2p - 3 = 0$, $p = 3$ or $p = -1$. HA · 6

$p = 3$: $y = \mathbf{6}x - \mathbf{9}$, touching at $(3, 9)$; $p = -1$: $y = \mathbf{-2}x - \mathbf{1}$, touching at $(-1, 1)$.

b. At (p, p^2) the tangent's slope is $2p$. If $p \neq 0$, the normal is $y - p^2 = -\frac{1}{2p}(x - p)$; through $\left(0, \frac{5}{2}\right)$: $\frac{5}{2} - p^2 = \frac{1}{2}$, $p^2 = 2$, $p = \pm\sqrt{2}$.

The normals: $y - 2 = \mp\frac{1}{2\sqrt{2}}(x \mp \sqrt{2})$, that is $x + \mathbf{2}\sqrt{\mathbf{2}}\,y = \mathbf{5}\sqrt{\mathbf{2}}$ at $(\sqrt{2}, 2)$ and $x - \mathbf{2}\sqrt{\mathbf{2}}\,y = \mathbf{-5}\sqrt{\mathbf{2}}$ at $(-\sqrt{2}, 2)$.

If $p = 0$, the tangent is horizontal and the normal is the line $x = \mathbf{0}$, which passes through $\left(0, \frac{5}{2}\right)$ too.

So there are **three** normals. (The tempting slip: writing the normal's slope as $-\frac{1}{2p}$ for every p loses $p = 0$.)

MARKING SCHEME

- ✓ The tangent at (p, p^2) made to pass through $(1, -3)$: $p^2 - 2p - 3 = 0$.
- ✓ $y = 6x - 9$ at $(3, 9)$ and $y = -2x - 1$ at $(-1, 1)$.
- ✓ The normal at (p, p^2) through the point: $p^2 = 2$.
- ✓ $p = \pm\sqrt{2}$.
- ✓ The two normals there.
- ✓ The normal $x = 0$ at $p = 0$: three in all.

4 A drone going up

ANSWER A · 5

Tests: relate first, differentiate, then substitute; the rate of an angle; where a rate is greatest.

Let the height be h m, the angle θ and the distance s m: $\tan\theta = \frac{h}{40}$ and $s^2 = 1600 + h^2$ at every instant.

a. $\sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{40}\frac{dh}{dt}$. At $h = 30$, $\tan\theta = \frac{3}{4}$, $\sec^2\theta = \frac{25}{16}$: $\frac{d\theta}{dt} = \frac{16}{25}\cdot\frac{3}{40} = \mathbf{0.048\ rad/s}$.

b. $2s\frac{ds}{dt} = 2h\frac{dh}{dt}$: $\frac{ds}{dt} = \frac{30}{50}\times 3 = \mathbf{1.8\ m/s}$.

c. She substituted before differentiating. 50 m is the distance at one instant only; the distance is changing, so its equation must be differentiated first (b) and the numbers put in afterwards.

d. $\frac{d\theta}{dt} = \frac{40}{1600 + h^2}\times 3 = \frac{120}{1600 + h^2}$, which is greatest when $h = 0$: **at take-off**, 0.075 rad/s. Near the ground each meter up turns her line of sight a lot; high up the same meter turns it very little.

MARKING SCHEME

- ✓ $\tan\theta = \frac{h}{40}$ differentiated with respect to t.
- ✓ $\frac{d\theta}{dt} = 0.048$ rad/s.
- ✓ $\frac{ds}{dt} = 1.8$ m/s.
- ✓ c: substituted before differentiating (50 m holds only at one instant).
- ✓ d: $\frac{d\theta}{dt} = \frac{120}{1600 + h^2}$, greatest at $h = 0$.

5 One derivative, two functions

ANSWER

Tests: the chain rule with an inverse hyperbolic function; the same derivative on two domains belonging to two different functions ($\tanh^{-1}$ inside $(-1, 1)$, $\coth^{-1}$ outside).

a. With $u = \frac{2x}{1 + x^2}$: $\frac{du}{dx} = \frac{2(1 + x^2) - 2x\cdot 2x}{(1 + x^2)^2} = \frac{2(1 - x^2)}{(1 + x^2)^2}$, and

$1 - u^2 = \frac{(1 + x^2)^2 - 4x^2}{(1 + x^2)^2} = \frac{(1 - x^2)^2}{(1 + x^2)^2}$.

$\frac{dy}{dx} = \frac{1}{1 - u^2}\frac{du}{dx} = \frac{(1 + x^2)^2}{(1 - x^2)^2}\cdot\frac{2(1 - x^2)}{(1 + x^2)^2} = \frac{2}{1 - x^2}$ ✓

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b. $\tanh^{-1} x$ is defined only for $|x| < 1$, so $2\tanh^{-1} x$ means nothing for $|x| > 1$, while y is defined there ($\left|\frac{2x}{1+x^2}\right| < 1$ for $x \neq \pm 1$). For $|x| > 1$ the function with derivative $\frac{2}{1-x^2}$ is $2\coth^{-1} x$ (check: at $x = 3$, $\tanh^{-1}0.6 = 0.693 = 2\coth^{-1}3$). HA · 4

c. At $x = 1$, $\frac{2x}{1+x^2} = 1$, and $\tanh^{-1} 1$ is not defined ($\frac{1}{2}\ln\frac{2}{0}$).

MARKING SCHEME

✓ $\frac{du}{dx}$ and $1 - u^2$ found.

✓ Simplified to $\frac{2}{1-x^2}$.

✓ $\tanh^{-1} x$ doesn't exist for $|x| > 1$; there $y = 2\coth^{-1} x$.

✓ At $x = 1$ the inside is 1, where $\tanh^{-1}$ is not defined.

6 A hanging cable

ANSWER A · 5

Tests: $\cosh^2 - \sinh^2 = 1$ used inside a derivative; what a differential measures (the rise along the tangent) and when it fails.

a. $\frac{dy}{dx} = \sinh\frac{x}{a}$, $\frac{d^2y}{dx^2} = \frac{1}{a}\cosh\frac{x}{a}$. Since $\cosh^2 - \sinh^2 = 1$ and $\cosh > 0$:
$\cosh\frac{x}{a} = \sqrt{1 + \sinh^2\frac{x}{a}} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ ✓

b. $\text{Sag} = 200\cosh\frac{50}{200} - 200 = 200(\cosh 0.25 - 1) = 200 \times 0.0314 = \mathbf{6.28\ m}$.

c. At the lowest point the tangent is horizontal, so the rise along it is 0 for any dx: the differential gives only the tangent's rise. The sag comes entirely from the curve bending away from its tangent, which is what dy leaves out; and 50 m is not a small dx either.

MARKING SCHEME

✓ $y' = \sinh\frac{x}{a}$, $y'' = \frac{1}{a}\cosh\frac{x}{a}$.

✓ $\cosh = \sqrt{1 + \sinh^2}$ to the result.

✓ Sag $= 200(\cosh 0.25 - 1)$.

✓ 6.28 m.

✓ The tangent at the lowest point is flat: dy gives the tangent's rise, 0, not the curve's.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	10	19	**29**

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