Unit 12 · Derivatives

ANSWER KEY · FOR TEACHERS

Full marks 34

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

1. $\sinh x \to$ **f.** $\dfrac{e^x - e^{-x}}{2}$ — K · 3
2. $\cosh x \to$ **a.** $\dfrac{e^x + e^{-x}}{2}$
3. $\tanh^{-1} x$, for $|x| < 1 \to$ **d.** $\dfrac{1}{2}\ln\dfrac{1+x}{1-x}$
4. the differential $dy \to$ **e.** $f'(x)\,dx$
5. the slope of the normal where the tangent's slope is $m \to$ **g.** $-\dfrac{1}{m}$
6. the forms L'Hospital's rule works on $\to$ **b.** $\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$

Each carries 0.5 mark.

2 True or false

i. ✗ **False**: $\dfrac{d}{dx}(\cosh x) = \sinh x$, with no minus: $\dfrac{d}{dx}\dfrac{e^x + e^{-x}}{2} = \dfrac{e^x - e^{-x}}{2}$. — U · 3

ii. ✓ **True**

iii. ✗ **False**: It differentiates the top and the bottom separately: $\lim\dfrac{f'(x)}{g'(x)}$.

iv. ✗ **False**: dy is the rise along the tangent; it is only close to Δy when dx is small.

v. ✓ **True**

vi. ✗ **False**: Only for $-1 < x < 1$, since $\tanh x$ always lies between -1 and 1.

Each carries 0.5 mark.

3 Multiple choice

i. **a.** $\cosh x$ — K · 1

$\dfrac{d}{dx}\dfrac{e^x - e^{-x}}{2} = \dfrac{e^x + e^{-x}}{2} = \cosh x.$

ii. **b.** $\dfrac{1}{2}$ — U · 1

$\dfrac{0}{0}$: $\lim\dfrac{\sin x}{2x}$, again $\dfrac{0}{0}$: $\lim\dfrac{\cos x}{2} = \dfrac{1}{2}$. 1 forgets the 2 from $(x^2)'$.

iii. **b.** $\dfrac{2}{\sqrt{4x^2 - 1}}$ — U · 1

$\dfrac{1}{\sqrt{(2x)^2 - 1}} \times 2$, by the chain rule. The first forgets the 2; the third is $\sinh^{-1}$'s.

iv. **b.** $y = -2x - 1$ — A · 1

$y' = 2x - 4 = -2$ at $x = 1$: $y + 3 = -2(x - 1)$, $y = -2x - 1$. $2y = x - 7$ is the normal.

v. **b.** $(1, e)$ — HA · 1

At (a, e^a) the tangent is $y - e^a = e^a(x - a)$. Through $(0, 0)$: $-e^a = -ae^a$, so $a = 1$. The tangent at $(1, 1)$, $y = x + 1$, misses the origin.

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4 Short answers

a. MARKING SCHEME — U · 2

✓ No: the form is $\frac{0}{1}$, not indeterminate.

✓ The limit is 0.

b. MARKING SCHEME — A · 2

✓ $dy = \frac{1}{2\sqrt{49}} \times 1 = \frac{1}{14}$.

✓ $\sqrt{50} \approx 7.0714$.

c. MARKING SCHEME — A · 3

✓ The first use is right ($\frac{0}{0}$).

✓ The second is wrong: $\frac{e^x}{2x}$ is of the form $\frac{1}{0}$, not indeterminate.

✓ The limit does not exist ($+\infty$ from the right, $-\infty$ from the left).

d. ANSWER — A · 5

$1 + 2 + 4 = 7$ ✓.

Differentiate implicitly: $2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$, so $\frac{dy}{dx} = -\frac{2x+y}{x+2y} = -\frac{4}{5}$ at $(1, 2)$.

Tangent: $y - 2 = -\frac{4}{5}(x - 1)$, that is $\mathbf{4x + 5y = 14}$.

Normal (slope $\frac{5}{4}$): $y - 2 = \frac{5}{4}(x - 1)$, that is $\mathbf{5x - 4y + 3 = 0}$.

MARKING SCHEME

✓ $(1, 2)$ checked on the curve.

✓ Implicit differentiation, $\frac{dy}{dx} = -\frac{2x+y}{x+2y}$.

✓ Slope $-\frac{4}{5}$ at the point.

✓ Tangent $4x + 5y = 14$.

✓ Normal $5x - 4y + 3 = 0$.

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5 Long answer

ANSWER A · 8

a. $\frac{d}{dx}\ln\left(x+\sqrt{x^2+1}\right) = \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{x+\sqrt{x^2+1}}\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}$ ✓

b. $y_1 = \frac{2\sinh^{-1}x}{\sqrt{1+x^2}}$, so $(1+x^2){y_1}^2 = 4(\sinh^{-1}x)^2 = 4y$.

Differentiate: $(1+x^2)\cdot 2y_1y_2 + 2x{y_1}^2 = 4y_1$. Divide by $2y_1$: $(1+x^2)y_2 + xy_1 = 2$ ✓.

c. At depth h the water's radius is $r = \frac{h}{2}$ (similar triangles, 2 m across 4 m), so $V = \frac{1}{3}\pi r^2 h = \frac{\pi h^3}{12}$.

$\frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt}$: $0.5 = \frac{\pi}{4}\frac{dh}{dt}$ at $h = 1$, so $\frac{dh}{dt} = \frac{2}{\pi} \approx$ **0.64 m a minute**.

What it means: the level rises fast at first, when the narrow bottom fills, and more slowly as the cone widens.

MARKING SCHEME

- ✓ The chain rule on $\ln\left(x+\sqrt{x^2+1}\right)$.
- ✓ The derivative of $\sqrt{x^2+1}$ is $\frac{x}{\sqrt{x^2+1}}$.
- ✓ Simplified to $\frac{1}{\sqrt{x^2+1}}$.
- ✓ $(1+x^2){y_1}^2 = 4y$.
- ✓ Differentiated with the product rule.
- ✓ Divided by $2y_1$ to the result.
- ✓ $V = \frac{\pi h^3}{12}$ from $r = \frac{h}{2}$.
- ✓ $\frac{dh}{dt} = \frac{2}{\pi} \approx 0.64$ m a minute.

6 Think

ANSWER HA · 3

$\Delta A = (x+dx)^2 - x^2 = 2x\,dx + (dx)^2$. In the drawing, the new area round the old square is two strips, each x long and dx wide ($2x\,dx$, which is dA), and a small corner square dx by dx.

dA leaves out the corner, $(dx)^2$: tiny when dx is small (for $dx = 0.01$, a ten-thousandth), which is why $dA \approx \Delta A$.

MARKING SCHEME

- ✓ The drawing: two strips x by dx and a corner dx by dx.
- ✓ $\Delta A = 2x\,dx + (dx)^2$, and $dA = 2x\,dx$ is the two strips.
- ✓ dA leaves out the corner $(dx)^2$, very small when dx is small.

7 Activity: How good is dy?

(not marked)

WHAT STUDENTS SHOULD FIND

Δy: 7.625, 1.261, 0.120601; dy: 6, 1.2, 0.12; difference 1.625, 0.061, 0.000601. Each time dx is made ten times smaller, the difference becomes about a hundred times smaller (it is $6(dx)^2 + (dx)^3$), so dy is a good stand-in for Δy when dx is small.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	4	7	19	4	**34**

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