Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 How many bulbs to test?

ANSWER HA · 6

Tests: a selection without putting back from a small lot is not independent; choosing n from a condition; the trade-off between letting bad boxes through and turning good ones away.

a. Accepted means all n come from the 16 good bulbs: $P = \dfrac{C(16, n)}{C(20, n)}$.

$n = 5$: $\dfrac{4368}{15\,504} = 0.282$; $n = 6$: $\dfrac{8008}{38\,760} = 0.207$; $n = 7$: $\dfrac{11\,440}{77\,520} = 0.148$.

The least is $n = \mathbf{7}$.

b. Bibek's chance is $(0.8)^n$: $(0.8)^7 = 0.210$, still above 0.2, and $(0.8)^8 = 0.168$, so he says **8**. Without putting back, every good bulb taken out leaves the faulty ones a larger share of what is left ($\dfrac{16}{20} \cdot \dfrac{15}{19} \cdot \dfrac{14}{18} \cdots$), so the true chance of all-good is smaller than $(0.8)^n$: his method makes her test one bulb too many. In a box of 20 the draws are clearly dependent.

c. $\dfrac{C(19, 7)}{C(20, 7)} = \dfrac{13}{20} = \mathbf{0.65}$: a box with only one faulty bulb is turned away 35% of the time. Testing more bulbs catches more bad boxes but also rejects more nearly good ones (and tests use time and bulbs): the owner has to weigh the two.

MARKING SCHEME

- ✓ $P(\text{accepted}) = \dfrac{C(16, n)}{C(20, n)}$.
- ✓ Values worked for $n = 6$ (0.207) and $n = 7$ (0.148).
- ✓ $n = 7$.
- ✓ Bibek's $(0.8)^n$ gives $n = 8$ (0.210 at 7).
- ✓ Draws without putting back from 20 are dependent; his chance is too large.
- ✓ One faulty: $\dfrac{13}{20} = 0.65$ accepted, so good boxes are often rejected: a trade-off.

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2 What a positive test means

ANSWER HA · 5

Tests: $P(\text{has} \mid \text{positive})$ against $P(\text{positive} \mid \text{has})$, worked by counting inside the condition; why a rare condition makes most positives false.

a. Has the condition: $2\% \times 5000 = 100$; positive $90\% \times 100 = 90$, negative 10. Hasn't: 4,900; positive $5\% \times 4900 = 245$, negative 4,655.

	Positive	Negative	Total
Has the condition	90	10	100
Hasn't	245	4655	4900
Total	335	4665	5000

b. $P(\text{positive}) = \frac{335}{5000} = \mathbf{0.067}$.

c. Given positive, count among the 335 only: $P(\text{has} \mid \text{positive}) = \frac{90}{335} = \frac{18}{67} \approx \mathbf{0.269}$.

d. 90% is $P(\text{positive} \mid \text{has})$, the chance of the other condition. She needs $P(\text{has} \mid \text{positive})$, whose total is the 335 positives, and most of those (245) come from the 4,900 who don't have it, because they are so many more. So a positive raises her chance from 2% to about 27%, not to 90%: a second test is needed. (The tempting answer: 0.9.)

MARKING SCHEME

✓ The table: 90, 10, 245, 4655.

✓ $P(\text{positive}) = 0.067$.

✓ $P(\text{has} \mid \text{positive}) = \frac{90}{335} \approx 0.269$.

✓ 90% is $P(\text{positive} \mid \text{has})$, the reverse condition.

✓ Most positives come from the many students without it.

3 Independent is not the same as separate

ANSWER HA · 4

Tests: independence as $P(A \cap B) = P(A)P(B)$, not as "no overlap"; working with complements.

a. $P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A)P(B) = (1 - P(A))(1 - P(B)) = P(\overline{A})P(\overline{B})$. ✓

b. Mutually exclusive: $P(A \cap B) = 0$. But $P(A)P(B) > 0$, so $P(A \cap B) \neq P(A)P(B)$: dependent. (Knowing A happened makes B impossible.)

c. $P(A) = \frac{1}{2}$, $P(B) = \frac{4}{6} = \frac{2}{3}$, $A \cap B = \{2, 4\}$, $P = \frac{1}{3} = \frac{1}{2} \cdot \frac{2}{3}$: **independent**, though they overlap.

Odd and more than 4 is $\overline{A} \cap \overline{B}$: $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$. Listing: only 5, so $\frac{1}{6}$. ✓

MARKING SCHEME

✓ $P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)$ expanded and factored to $P(\overline{A})P(\overline{B})$.

✓ $P(A \cap B) = 0 \neq P(A)P(B)$.

✓ A and B independent: $\frac{1}{3} = \frac{1}{2} \cdot \frac{2}{3}$.

✓ $\frac{1}{6}$, checked: only the 5.

4 The third egg

ANSWER

Tests: an earlier draw that isn't seen doesn't change the chance; "the third is cracked" against "the first cracked one is the third"; a conditional probability inside a selection.

a. Before the third draw, 0, 1 or 2 cracked eggs may have gone:

$P(2 \text{ gone}) = \frac{3}{12} \cdot \frac{2}{11} = \frac{6}{132}$; $P(1 \text{ gone}) = 2 \cdot \frac{3}{12} \cdot \frac{9}{11} = \frac{54}{132}$; $P(0 \text{ gone}) = \frac{9}{12} \cdot \frac{8}{11} = \frac{72}{132}$.

$P(\text{third cracked}) = \frac{6}{132} \cdot \frac{1}{10} + \frac{54}{132} \cdot \frac{2}{10} + \frac{72}{132} \cdot \frac{3}{10} = \frac{6 + 108 + 216}{1320} = \frac{330}{1320} = \mathbf{\frac{1}{4}}$, the same as the first draw: not knowing the first two, you know nothing new about the third.

b. Whole, whole, cracked: $\frac{9}{12} \cdot \frac{8}{11} \cdot \frac{3}{10} = \frac{216}{1320} = \mathbf{\frac{9}{55}}$.

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c. Exactly one of the first three cracked: $\frac{C(3,1)C(9,2)}{C(12,3)} = \frac{108}{220} = \frac{27}{55}$. "Exactly one, and it is the third" is b's event. $P = \frac{9/55}{27/55} = \mathbf{\frac{1}{3}}$: given one cracked egg among three, each of the three places is equally likely. HA · 5

MARKING SCHEME

✓ The three cases for the first two draws, with their chances.

✓ $P(\text{third cracked}) = \frac{1}{4}$.

✓ $\frac{9}{12} \cdot \frac{8}{11} \cdot \frac{3}{10} = \frac{9}{55}$.

✓ $P(\text{exactly one in three}) = \frac{27}{55}$.

✓ $\frac{1}{3}$.

5 Make your own

ANSWER HA · 3

Tests: what conditioning does to a chance; independence as "knowing A doesn't change B".

$P(B) = \frac{1}{2}$. For example:

$A = \{4, 5, 6\}$: $P(B \mid A) = \frac{2}{3} > \frac{1}{2}$.

$A = \{1, 2, 3\}$: $P(B \mid A) = \frac{1}{3} < \frac{1}{2}$.

$A = \{1, 2, 3, 4\}$: $P(B \mid A) = \frac{2}{4} = \frac{1}{2}$: this A is independent of B. (Any A with as many even faces as odd ones works.)

MARKING SCHEME

✓ An A with $P(B \mid A) > \frac{1}{2}$, worked.

✓ An A with $P(B \mid A) < \frac{1}{2}$, worked.

✓ An A with $P(B \mid A) = \frac{1}{2}$, named as the independent one.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	0	23	**23**

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