Unit 9 · Product of Vectors

HARDER PRACTICE QUESTIONS

Full marks 25

Write your answers on separate paper. Show your working.

1 One sine, two angles [5]

$|\vec{a}| = 2$, $|\vec{b}| = 5$ and $|\vec{a} \times \vec{b}| = 8$.

a. Show that $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$ for any two vectors.

b. Find the possible values of $\vec{a} \cdot \vec{b}$ and of the angle θ between the vectors.

c. Pratik says $\theta = 53.1°$. If also $|\vec{a} + \vec{b}| = \sqrt{17}$, is he right?

2 A plot with a notch [6]

A surveyor's GPS gives the corners of a plot, in meters east and north of the corner A: $A(0, 0)$, $B(30, 0)$, $C(30, 20)$, $D(15, 8)$, $E(0, 20)$. The side CDE bends in, making a notch.

a. Cut the plot into the triangles ABC, ACD, ADE from A, and work out $x_1y_2 - x_2y_1$ for each pair of position vectors in turn. What is odd about one of them?

b. Show that half the sum of the three values is the plot's area, and check it another way.

c. Repeat a and b with the vectors measured from a survey pillar $O(10, -5)$ outside the plot, using the five triangles OAB, OBC, OCD, ODE, OEA.

d. Rojina says: "Areas can't be negative, so take every value as positive." What would she get from a, and why is she wrong?

3 The turning effect of a force [5]

A force $\vec{F} = 2\vec{i} - \vec{j} + 3\vec{k}$ newtons acts at the point $P(1, 2, 1)$ (in meters). Its moment (turning effect) about a point Q is $\overrightarrow{QP} \times \vec{F}$.

a. Find the moment about the origin, and its size.

b. Find the moment about $Q(0, 1, -1)$.

c. Puja says: "A force has one moment, whatever point you take it about." b shows she is wrong, but not about every pair of points. Find all the points Q about which the moment is the same as about the origin, and all those about which it is $\vec{0}$. Explain each.

4 Perpendicular diagonals in any quadrilateral [4]

$ABCD$ is any quadrilateral, not necessarily a parallelogram.

a. Show that $AB^2 + CD^2 - BC^2 - DA^2 = 2\overrightarrow{AC} \cdot \overrightarrow{DB}$.

b. Hence prove that the diagonals AC and BD are perpendicular exactly when $AB^2 + CD^2 = BC^2 + DA^2$.

c. Use b to show that a parallelogram has perpendicular diagonals exactly when it is a rhombus.

5 How far from a line in space? [5]

A straight power line runs through $A(1, 0, 2)$ in the direction $\vec{u} = 2\vec{i} + \vec{j} - 2\vec{k}$ (in meters). A drone hovers at $P(3, 4, 1)$.

a. Kiran says the drone is $\dfrac{\overrightarrow{AP} \cdot \vec{u}}{|\vec{u}|}$ meters from the line. Work out his number and say what it really measures.

b. Using $|\overrightarrow{AP} \times \vec{u}| = |\overrightarrow{AP}||\vec{u}|\sin\theta$, find the true distance from P to the line.

c. Check b by Pythagoras, and find the point of the line nearest the drone.

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