Unit 9 · Product of Vectors

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 25

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 One sine, two angles

ANSWER A · 5

Tests: the vector product fixes $\sin\theta$, which belongs to two angles; the scalar product decides.

a. $|\vec{a}\times\vec{b}|^2 + (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2(\sin^2\theta+\cos^2\theta) = |\vec{a}|^2|\vec{b}|^2$.

b. $(\vec{a}\cdot\vec{b})^2 = 100 - 64 = 36$, so $\vec{a}\cdot\vec{b} = \pm 6$: $\cos\theta = \pm 0.6$, $\theta \approx 53.1°$ or $126.9°$ (both have $\sin\theta = 0.8$).

c. $|\vec{a}+\vec{b}|^2 = 4 + 25 + 2\vec{a}\cdot\vec{b} = 17$, so $\vec{a}\cdot\vec{b} = -6$ and $\theta \approx 126.9°$. Pratik is wrong: he took the acute angle that $\sin\theta = 0.8$ gives.

MARKING SCHEME

- ✓ a. $\sin^2\theta + \cos^2\theta = 1$ used.
- ✓ b. $\vec{a}\cdot\vec{b} = \pm 6$.
- ✓ b. $53.1°$ or $126.9°$.
- ✓ c. $\vec{a}\cdot\vec{b} = -6$ from $|\vec{a}+\vec{b}|^2$.
- ✓ c. $\theta \approx 126.9°$: Mohan is wrong.

2 A plot with a notch

ANSWER HA · 6

Tests: the $\vec{k}$ part of a vector product as a signed area; why the sign is needed when the shape isn't convex.

a. ABC: (30, 0), (30, 20): $600 - 0 = 600$. ACD: (30, 20), (15, 8): $240 - 300 = -60$. ADE: (15, 8), (0, 20): $300 - 0 = 300$. The middle one is **negative**: going from $\overrightarrow{AC}$ to $\overrightarrow{AD}$ turns clockwise, because D lies inside the notch, to the right of AC.

b. $\frac{1}{2}(600 - 60 + 300) = 420$ m^2. Check: the rectangle $30 \times 20 = 600$, less the notch CDE, a triangle of base 30 and height $20 - 8 = 12$: $600 - 180 = 420$. ✓ Why the minus is right: ABC and ADE together cover the plot, but ABC also covers the part of ACD that lies in the notch, outside the plot, and the rest of ACD is covered by both. So the two count exactly ACD too much, 30 m^2, and its value, -60, takes it off.

c. From O: $\overrightarrow{OA} = (-10, 5)$, $\overrightarrow{OB} = (20, 5)$, $\overrightarrow{OC} = (20, 25)$, $\overrightarrow{OD} = (5, 13)$, $\overrightarrow{OE} = (-10, 25)$. The values, $x_1y_2 - x_2y_1$ for each pair in turn: OAB: $-50 - 100 = -150$; OBC: $500 - 100 = 400$; OCD: $260 - 125 = 135$; ODE: $125 + 130 = 255$; OEA: $-50 + 250 = 200$. Sum $-150 + 400 + 135 + 255 + 200 = 840$: area **420** m^2 again.

d. $\frac{1}{2}(600 + 60 + 300) = 480$ m^2, 60 m^2 too much: ACD, already counted too much by the other two, would be added instead of taken off. The sign is what makes the sum right for any shape and any starting point.

MARKING SCHEME

- ✓ a. 600, −60, 300.
- ✓ a. One is negative: D is on the other side of AC.
- ✓ b. 420 m^2, checked as $600 - 180$.
- ✓ c. The five values from O.
- ✓ c. Sum 840, area 420 m^2.
- ✓ d. 480 m^2 is wrong: the sign takes off what the other triangles count too much.

3 The turning effect of a force

ANSWER

Tests: the vector product as a physical quantity that depends on the point chosen; the distributive law and parallel vectors deciding exactly which points.

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a. $\overrightarrow{OP} \times \vec{F} = (2\times 3 - 1\times(-1))\vec{i} + (1\times 2 - 1\times 3)\vec{j} + (1\times(-1) - 2\times 2)\vec{k} = 7\vec{i} - \vec{j} - 5\vec{k}$ N m, of size $\sqrt{75} = 5\sqrt{3} \approx 8.66$ N m. **A · 5**

b. $\overrightarrow{QP} = (1, 1, 2)$: $\overrightarrow{QP} \times \vec{F} = (3+2)\vec{i} + (4-3)\vec{j} + (-1-2)\vec{k} = 5\vec{i} + \vec{j} - 3\vec{k}$ N m: a different moment.

c. With $\vec{q}$ the position vector of Q, $\overrightarrow{QP} \times \vec{F} = (\vec{p} - \vec{q}) \times \vec{F} = \vec{p} \times \vec{F} - \vec{q} \times \vec{F}$ (distributive). It equals the moment about the origin exactly when $\vec{q} \times \vec{F} = \vec{0}$, $\vec{q}$ parallel to $\vec{F}$: the points $t\vec{F} = (2t, -t, 3t)$, **the line through the origin along the force**. Sliding the point along the force's direction leaves the moment as it was. It is $\vec{0}$ exactly when $\overrightarrow{QP}$ is parallel to $\vec{F}$: **the force's own line of action**, $(1 + 2t, 2 - t, 1 + 3t)$, such as $(3, 1, 4)$ (a door pushed straight at its hinge doesn't turn). Taking $Q = P$ alone answers only half of this.

MARKING SCHEME

✓ a. $7\vec{i} - \vec{j} - 5\vec{k}$.
✓ a. $5\sqrt{3}$ N m.
✓ b. $5\vec{i} + \vec{j} - 3\vec{k}$.
✓ c. The same as about O exactly when $\vec{q} \times \vec{F} = \vec{0}$: the line $t\vec{F}$.
✓ c. $\vec{0}$ exactly when $\overrightarrow{QP} \parallel \vec{F}$: the line of action $P + t\vec{F}$.

4 Perpendicular diagonals in any quadrilateral

ANSWER **HA · 4**

Tests: a length squared as a vector times itself, and "exactly when" proved both ways, for a shape the unit's parallelogram proof doesn't cover.

a. With position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$: $AB^2 = |\vec{b} - \vec{a}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}$, and the same for the other sides. In $AB^2 + CD^2 - BC^2 - DA^2$ each of $|\vec{a}|^2$, $|\vec{b}|^2$, $|\vec{c}|^2$, $|\vec{d}|^2$ comes once with + and once with −, leaving $2(\vec{b} \cdot \vec{c} + \vec{d} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{c} \cdot \vec{d})$. And $2\overrightarrow{AC} \cdot \overrightarrow{DB} = 2(\vec{c} - \vec{a}) \cdot (\vec{b} - \vec{d}) = 2(\vec{c} \cdot \vec{b} - \vec{c} \cdot \vec{d} - \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{d})$, the same. ✓ (Taking A as the origin shortens it.)

b. If $AC \perp BD$, $\overrightarrow{AC} \cdot \overrightarrow{DB} = 0$, so by a, $AB^2 + CD^2 = BC^2 + DA^2$. Conversely, if the sums are equal, a gives $\overrightarrow{AC} \cdot \overrightarrow{DB} = 0$, and since the diagonals are not zero vectors, they are perpendicular.

c. In a parallelogram $AB = CD$ and $BC = DA$, so the condition of b reads $2AB^2 = 2BC^2$, that is $AB = BC$: all four sides equal, a rhombus. So its diagonals are perpendicular exactly when it is a rhombus (Example 5 of the unit, and its converse, from a different route). The tempting shortcut, "perpendicular diagonals make a rhombus", fails for a quadrilateral in general: a kite has them with $AB \neq BC$.

MARKING SCHEME

✓ a. Each side squared expanded as a vector times itself.
✓ a. The squares cancel, leaving $2\overrightarrow{AC} \cdot \overrightarrow{DB}$.
✓ b. Both directions argued from a.
✓ c. $AB = CD$, $BC = DA$ turn b into $AB = BC$: a rhombus.

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5 How far from a line in space?

ANSWER A · 5

Tests: projection (along the line) against distance (across it); the vector product giving $|\overrightarrow{AP}|\sin\theta$.

a. $\overrightarrow{AP} = (2, 4, -1)$, $\overrightarrow{AP} \cdot \vec{u} = 4 + 4 + 2 = 10$, $|\vec{u}| = 3$: $\frac{10}{3} \approx 3.33$ m. That is the projection of $\overrightarrow{AP}$ on the line: how far **along** the line, from A, the nearest point is, not the distance to the line.

b. The distance is $|\overrightarrow{AP}|\sin\theta = \frac{|\overrightarrow{AP} \times \vec{u}|}{|\vec{u}|}$.
$\overrightarrow{AP} \times \vec{u} = (4 \times (-2) - (-1) \times 1)\vec{i} + ((-1) \times 2 - 2 \times (-2))\vec{j} + (2 \times 1 - 4 \times 2)\vec{k} = -7\vec{i} + 2\vec{j} - 6\vec{k}$, of length $\sqrt{89}$. Distance $\frac{\sqrt{89}}{3} \approx 3.14$ m.

c. $|\overrightarrow{AP}|^2 = 21$; $21 - \left(\frac{10}{3}\right)^2 = \frac{189 - 100}{9} = \frac{89}{9}$, the square of b. ✓ The nearest point is $A + \frac{10}{9}\vec{u} = \left(\frac{29}{9}, \frac{10}{9}, -\frac{2}{9}\right)$ (the projection $\frac{10}{3}$ times the unit vector $\frac{\vec{u}}{3}$).

MARKING SCHEME

✓ a. $\frac{10}{3}$: the distance along the line, a projection.

✓ b. $\overrightarrow{AP} \times \vec{u} = -7\vec{i} + 2\vec{j} - 6\vec{k}$.

✓ b. $\frac{\sqrt{89}}{3} \approx 3.14$ m.

✓ c. Pythagoras: $21 - \frac{100}{9} = \frac{89}{9}$.

✓ c. The nearest point $\left(\frac{29}{9}, \frac{10}{9}, -\frac{2}{9}\right)$.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	15	10	**25**

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