Unit 9 · Product of Vectors

ANSWER KEY · FOR TEACHERS

Full marks 37

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

K · 4

1. scalar
2. perpendicular
3. projection
4. work
5. vector
6. parallel
7. parallelogram
8. right-hand

Each carries 0.5 mark.

2 True or false

U · 3

i. ✓ **True**

ii. ✗ **False**: $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$: the same length, the opposite direction.

iii. ✓ **True**

iv. ✗ **False**: They are **perpendicular**; parallel vectors have $\vec{a} \times \vec{b} = \vec{0}$.

v. ✗ **False**: It is the area of the **parallelogram**; the triangle has half of it.

vi. ✓ **True**

Each carries 0.5 mark.

3 Multiple choice

i. **a.** $\vec{i}$ — K · 1

In the order $\vec{i} \to \vec{j} \to \vec{k} \to \vec{i}$, each product of two in turn gives the next: $\vec{j} \times \vec{k} = \vec{i}$. $\vec{k} \times \vec{j} = -\vec{i}$. A vector product is a vector, never the number 0 or 1.

ii. **a.** 4 — U · 1

$2 \times 1 + (-1) \times 4 + 3 \times 2 = 2 - 4 + 6 = 4$. 12 forgets the minus sign; the third multiplies the components but doesn't add them.

iii. **c.** 60° — U · 1

$\cos\theta = \dfrac{0+1+0}{\sqrt{2}\sqrt{2}} = \dfrac{1}{2}$. They aren't perpendicular: they share the $\vec{j}$ part.

iv. **b.** $6\sqrt{3}$ — A · 1

$\cos\theta = \dfrac{6}{12} = \dfrac{1}{2}$, so $\sin\theta = \dfrac{\sqrt{3}}{2}$ and $|\vec{a} \times \vec{b}| = 12 \times \dfrac{\sqrt{3}}{2} = 6\sqrt{3}$. (Or $|\vec{a} \times \vec{b}|^2 = 144 - 36 = 108$.) 12 is $|\vec{a}||\vec{b}|$, true only at 90°.

v. **b.** $\vec{a}$ and $\vec{b}$ are perpendicular — HA · 1

Squaring: $|\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 = |\vec{a}|^2 - 2\vec{a} \cdot \vec{b} + |\vec{b}|^2$, so $\vec{a} \cdot \vec{b} = 0$. A parallelogram whose diagonals are equal is a rectangle.

4 Short answers

a. — U · 2

MARKING SCHEME

✓ The same length $|\vec{a}||\vec{b}|\sin\theta$.

✓ The right-hand rule reverses when the order is reversed.

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b. MARKING SCHEME U · 2

✓ Divide by $|\vec{a}|$, not $|\vec{b}|$.

✓ $\frac{11}{3}$.

c. MARKING SCHEME A · 2

✓ $\vec{s} = (3, 2, 1)$.

✓ 13 J.

d. MARKING SCHEME A · 3

✓ $\vec{a} \times \vec{b} = \vec{i} - \vec{j} + \vec{k}$.

✓ Its length $\sqrt{3}$.

✓ Both unit vectors, $\pm\frac{1}{\sqrt{3}}(\vec{i} - \vec{j} + \vec{k})$.

5 Long answers

a. ANSWER A · 5

Let O be the center, AB a diameter and P any other point of the circle, radius r. Take $\overrightarrow{OA} = \vec{a}$, so $\overrightarrow{OB} = -\vec{a}$, and $\overrightarrow{OP} = \vec{p}$, with $|\vec{a}| = |\vec{p}| = r$.
Then $\overrightarrow{AP} = \vec{p} - \vec{a}$ and $\overrightarrow{BP} = \vec{p} + \vec{a}$.
$\overrightarrow{AP} \cdot \overrightarrow{BP} = (\vec{p} - \vec{a}) \cdot (\vec{p} + \vec{a}) = |\vec{p}|^2 - |\vec{a}|^2 = r^2 - r^2 = 0$.
Neither vector is zero, so $AP \perp BP$: $\angle APB = 90°$. ∎

MARKING SCHEME

✓ The vectors $\vec{a}$, $-\vec{a}$, $\vec{p}$ from the center.

✓ $\overrightarrow{AP} = \vec{p} - \vec{a}$, $\overrightarrow{BP} = \vec{p} + \vec{a}$.

✓ The scalar product $|\vec{p}|^2 - |\vec{a}|^2$.

✓ Both lengths r, so it is 0.

✓ So the angle is 90°.

b. ANSWER A · 8

i. $\overrightarrow{AB} = 2\vec{i} + \vec{j} + 2\vec{k}$, length 3; $\overrightarrow{AC} = \vec{i} + 2\vec{j} - \vec{k}$, length $\sqrt{6}$.

ii. $\overrightarrow{AB} \cdot \overrightarrow{AC} = 2 + 2 - 2 = 2$, so $\cos A = \frac{2}{3\sqrt{6}} \approx 0.272$, $A \approx 74.2°$.

iii. $\overrightarrow{AB} \times \overrightarrow{AC} = (1 \times (-1) - 2 \times 2)\vec{i} + (2 \times 1 - 2 \times (-1))\vec{j} + (2 \times 2 - 1 \times 1)\vec{k} = -5\vec{i} + 4\vec{j} + 3\vec{k}$, of length $\sqrt{50} = 5\sqrt{2}$. Area $= \frac{5\sqrt{2}}{2} \approx 3.54$ square units.

iv. Area $= \frac{1}{2} \times AB \times h$, so $h = \frac{5\sqrt{2}}{3} \approx 2.36$.

MARKING SCHEME

✓ i. $\overrightarrow{AB}$, $\overrightarrow{AC}$.

✓ i. Lengths 3 and $\sqrt{6}$.

✓ ii. The scalar product 2.

✓ ii. $A \approx 74.2°$.

✓ iii. $-5\vec{i} + 4\vec{j} + 3\vec{k}$.

✓ iii. Length $5\sqrt{2}$.

✓ iii. Area $\frac{5\sqrt{2}}{2}$.

✓ iv. $h = \frac{5\sqrt{2}}{3}$.

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6 Think

HA · 3

ANSWER

$\vec{c} = -\vec{a} - \vec{b}$, so $\vec{b} \times \vec{c} = -\vec{b} \times \vec{a} - \vec{b} \times \vec{b} = \vec{a} \times \vec{b}$ (since $\vec{b} \times \vec{b} = \vec{0}$ and $-\vec{b} \times \vec{a} = \vec{a} \times \vec{b}$). In the same way $\vec{c} \times \vec{a} = -\vec{a} \times \vec{a} - \vec{b} \times \vec{a} = \vec{a} \times \vec{b}$.

The three vectors go round a triangle; each product's length is twice its area, so $ab \sin C = bc \sin A = ca \sin B$: dividing by abc gives the sine law.

MARKING SCHEME

- ✓ $\vec{c} = -\vec{a} - \vec{b}$ substituted.
- ✓ $\vec{b} \times \vec{b} = \vec{0}$ and the order reversed with a minus.
- ✓ Each is twice the triangle's area (the sine law).

7 Activity: The area of a field from its corners

(not marked)

WHAT STUDENTS SHOULD FIND

$30 - 7 = 23$, area 11.5; $42 - 10 = 32$, area 16; the field 27.5 square units. A count of whole and half squares gives about the same. This is how a surveyor finds a plot's area from GPS readings.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	5	9	19	4	**37**

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