Unit 8 · Conic Section

HARDER PRACTICE QUESTIONS

Full marks 28

Write your answers on separate paper. Show your working.

1 Two tangents at a right angle [6]

From a point $P(h, k)$ outside the circle $x^2 + y^2 = a^2$, two tangents are drawn.

a. Show that the slopes m of the tangents satisfy $(h^2 - a^2)m^2 - 2hkm + k^2 - a^2 = 0$.

b. Without solving it, show that the tangents are at right angles exactly when $h^2 + k^2 = 2a^2$. What is the set of all such points P?

c. Check b with $P(7, 1)$ and $x^2 + y^2 = 25$, whose tangents from P are $3x + 4y = 25$ and $4x - 3y = 25$.

2 Where the receiver goes [5]

A satellite dish is a paraboloid: every cut through its axis is a parabola. The dish is 1.2 m across and 15 cm deep at the middle.

a. Take the vertex at the origin and the axis along the X-axis, so a cut is $y^2 = 4ax$. Find a, and say how far from the vertex the receiver must be placed.

b. Yogesh says: "Put the receiver level with the rim, 15 cm out." Use the reflecting property to explain what would happen.

c. A dish maker wants the receiver exactly level with the rim. How deep must a 1.2 m dish be? Show that for any such dish the depth is a quarter of the width.

3 A whispering gallery [6]

A hall is to be built with an elliptical floor plan, 20 m long. Sound from one focus bounces off the wall and passes through the other focus, so a whisper at one is heard at the other. The two whisper spots, the foci, are to be 2 m from the end walls.

a. Taking the center at the origin and the long axis along the X-axis, find how wide the hall must be, and write the equation of its wall.

b. A second hall is the same shape, turned so that its whisper spots are at $(0, \pm 8)$. Amrita writes its wall as $\frac{x^2}{100} + \frac{y^2}{36} = 1$. Find her mistake and the right equation.

c. In the first hall, Bibek stands against the wall 5 m from the whisper spot $(8, 0)$. How far is he from the other spot, and where on the wall is he standing?

4 Finding a sound by its timing [6]

Two listening posts, $A(-5, 0)$ and $B(5, 0)$ (in km), hear a blast. B hears it 2 seconds before A. Sound travels at 340 m/s.

a. How much farther is the blast from A than from B? Explain why the blast lies on one branch of a hyperbola with foci A and B, and which branch.

b. Find the equation of the hyperbola.

c. Far from the posts, the branch runs close to an asymptote. Find the angle the asymptote makes with AB, and say what this tells the listeners about the blast's direction.

d. What would a third post be needed for?

5 Tangents from the directrix [5]

a. Show that the tangents to $y^2 = 4ax$ from a point (h, k) have slopes m with $hm^2 - km + a = 0$.

(Use the tangent $y = mx + \frac{a}{m}$.)

b. Prove that the two tangents are at right angles exactly when (h, k) lies on the directrix.

c. For $y^2 = 8x$ and the point $(-2, 3)$, find both tangents and their points of contact, and show that the chord joining the points of contact passes through the focus.