Unit 8 · Conic Section

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 28

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Two tangents at a right angle

ANSWER HA · 6

Tests: the condition of tangency used to make a quadratic in the slope, then read through its roots' product, not solved.

a. A line through P with slope m is $mx - y + (k - mh) = 0$. It touches when its distance from O is a: $\frac{|k - mh|}{\sqrt{m^2 + 1}} = a$, so $(k - mh)^2 = a^2(m^2 + 1)$, that is $(h^2 - a^2)m^2 - 2hkm + k^2 - a^2 = 0$.

b. The two roots are the two slopes, and their product is $\frac{k^2 - a^2}{h^2 - a^2}$. The tangents are perpendicular exactly when this is -1: $k^2 - a^2 = a^2 - h^2$, that is $h^2 + k^2 = 2a^2$. So P lies on the circle $x^2 + y^2 = 2a^2$, with the same center and radius $\sqrt{2}$ times as large. (If $h^2 = a^2$, one tangent is upright, $x = h$, and has no slope; the two are then at right angles only when the other is level, which needs $k^2 = a^2$: a point of the same circle.)

c. $49 + 1 = 50 = 2 \times 25$, and the slopes $-\frac{3}{4}$ and $\frac{4}{3}$ multiply to -1. ✓ (The quadratic: $24m^2 - 14m - 24 = 0$, whose product of roots is -1.)

MARKING SCHEME

✓ a. The distance from the center equals a.

✓ a. The quadratic in m.

✓ b. The product of the roots $\frac{k^2 - a^2}{h^2 - a^2}$.

✓ b. Product -1 gives $h^2 + k^2 = 2a^2$.

✓ b. The circle $x^2 + y^2 = 2a^2$.

✓ c. $50 = 2 \times 25$ and the slopes' product -1.

2 Where the receiver goes

ANSWER A · 5

Tests: the focus found from a dish's real measurements; the focus is not at the rim or the depth; a design condition turned into an equation.

a. The rim's edge is at $(0.15, 0.6)$: $0.36 = 4a \times 0.15$, so $a = 0.6$ m. The receiver goes at the focus $(a, 0)$, **60 cm** from the vertex, 45 cm beyond the rim's level.

b. Rays coming in parallel to the axis are all reflected through the focus, 60 cm out. At 15 cm they haven't met yet: they are still spread across the dish's mouth, and a small receiver there would catch only the few that hit it, so the signal would be weak. (The tempting answer puts the focus at the depth, 15 cm, or on the rim; neither is the focus of this dish.)

c. Let the dish be w wide and d deep: the rim is at $\left(d, \frac{w}{2}\right)$, so $\frac{w^2}{4} = 4ad$. Level with the rim means the focus is at $x = d$, $a = d$: $\frac{w^2}{4} = 4d^2$, $w = 4d$, the depth a quarter of the width. For $w = 1.2$ m, $d = 0.3$ m (check: $0.36 = 4 \times 0.3 \times 0.3$). A shallower dish, like the first, has its focus out beyond the rim; a deeper one, inside the bowl.

MARKING SCHEME

✓ a. $(0.15, 0.6)$ on $y^2 = 4ax$.

✓ a. $a = 0.6$ m: 60 cm from the vertex.

✓ b. Parallel rays meet only at the focus.

✓ c. $a = d$ put into $\frac{w^2}{4} = 4ad$: 30 cm deep.

✓ c. $w = 4d$ for every such dish.

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3 A whispering gallery

ANSWER A · 6

Tests: a, b and the foci from a real shape, worked backwards; which axis is the major one; the sum of the focal distances and $PS = a - ex$ used together.

a. $2a = 20$, $a = 10$; the foci are 2 m from the ends, at $(\pm 8, 0)$: $ae = 8$. $b^2 = a^2 - (ae)^2 = 100 - 64 = 36$, $b = 6$: the hall is **12 m wide**, and the wall is $\mathbf{\frac{x^2}{100} + \frac{y^2}{36} = 1}$.

b. Her equation has the larger number under x^2, so its foci are on the X-axis, at $(\pm 8, 0)$. With the foci at $(0, \pm 8)$ the long axis is the Y-axis: the semi-axes are 10 along y and 6 along x: $\mathbf{\frac{x^2}{36} + \frac{y^2}{100} = 1}$.

c. $PS + PS' = 2a = 20$, so he is **15 m** from the other spot. $e = 0.8$, and his distance from $S(8, 0)$ is $a - ex = 10 - 0.8x = 5$: $x = 6.25$; then $y^2 = 36\left(1 - \frac{6.25^2}{100}\right) = 21.94$, $y \approx \pm 4.68$. He is at **(6.25, 4.68) or (6.25, −4.68)**. (Check: $\sqrt{1.75^2 + 4.68^2} \approx 5.0$.) The tempting answer takes a point 5 m from the spot along the long axis, (3, 0) or (13, 0), neither of them on the wall.

MARKING SCHEME

- ✓ a. $ae = 8$, $b = 6$: 12 m wide.
- ✓ a. $\frac{x^2}{100} + \frac{y^2}{36} = 1$.
- ✓ b. The major axis is the Y-axis.
- ✓ b. $\frac{x^2}{36} + \frac{y^2}{100} = 1$.
- ✓ c. 15 m, since $PS + PS' = 2a$.
- ✓ c. $10 - 0.8x = 5$: $(6.25, \pm 4.68)$.

4 Finding a sound by its timing

ANSWER A · 6

Tests: the hyperbola's definition (a fixed difference of distances) in a real setting; what a, b and the asymptotes mean there.

a. $340 \times 2 = 680$ m $= 0.68$ km farther from A. The points whose distances from two fixed points differ by a fixed amount, less than the distance between them, form a hyperbola with those foci. Nearer to B: the right-hand branch, $x > 0$.

b. $2a = 0.68$, $a = 0.34$, $a^2 = 0.1156$; $ae = 5$, $b^2 = 25 - 0.1156 = 24.8844$: $\mathbf{\frac{x^2}{0.1156} - \frac{y^2}{24.8844} = 1}$, $x > 0$.

c. Slope $\frac{b}{a} = \frac{4.988}{0.34} \approx 14.7$: about **86**° to AB. Far away, the blast is almost straight out from the middle of AB, tipped about 4° towards B (either side of the line AB: the hyperbola can't tell north from south).

d. One time difference gives only a curve. A third post gives a second difference and a second hyperbola, and the blast is where they cross.

MARKING SCHEME

- ✓ a. 0.68 km.
- ✓ a. A fixed difference of distances: a hyperbola, the branch nearer B.
- ✓ b. $a = 0.34$, $b^2 = 24.8844$.
- ✓ b. The equation.
- ✓ c. About 86° to AB, so nearly perpendicular to the base from its middle.
- ✓ d. A second hyperbola, to cross the first.

5 Tangents from the directrix

ANSWER

Tests: the tangent $y = mx + \frac{a}{m}$ used through a point; the product of the roots, read as a condition; the points of contact, and the chord they make.

a. The tangent passes through (h, k): $k = mh + \frac{a}{m}$; times m: $hm^2 - km + a = 0$.

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b. The product of the two slopes is $\frac{a}{h}$. It is -1 exactly when $h = -a$: the point is on the directrix $x = -a$, whatever k is.

c. $a = 2$, $h = -2$ (on the directrix $x = -2$): $-2m^2 - 3m + 2 = 0$, $2m^2 + 3m - 2 = 0$, $(2m - 1)(m + 2) = 0$: $m = \frac{1}{2}$ or -2. Tangents $y = \frac{1}{2}x + 4$ and $y = -2x - 1$, at right angles as b says (both pass through $(-2, 3)$). The tangent of slope m touches at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$: **(8, 8)** and $\left(\mathbf{\frac{1}{2}, -2}\right)$ (check: $64 = 8 \times 8$, $4 = 8 \times \frac{1}{2}$). The chord through them has slope $\frac{8 + 2}{8 - 0.5} = \frac{4}{3}$: $y - 8 = \frac{4}{3}(x - 8)$, which meets $y = 0$ at $x = 2$, the focus $(2, 0)$. ✓ (It holds for every point of the directrix: the chord of contact from a point of the directrix is a focal chord.)

MARKING SCHEME

✓ a. $k = mh + \frac{a}{m}$ gives the quadratic.

✓ b. The product $\frac{a}{h}$.

✓ b. -1 exactly when $h = -a$, the directrix.

✓ c. The tangents $y = \frac{1}{2}x + 4$, $y = -2x - 1$, touching at $(8, 8)$ and $\left(\frac{1}{2}, -2\right)$.

✓ c. The chord through them meets the axis at the focus $(2, 0)$.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	17	11	**28**

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