Unit 7 · Solution of Triangle

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 27

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 When the sine law misleads

ANSWER A · 5

Tests: $\sin\theta = \sin(180^\circ - \theta)$, so the sine law can't tell an obtuse angle from an acute one.

a. $a^2 = 9 + 64 - 48\cos 40^\circ \approx 73 - 36.77 = 36.23$, $a \approx 6.02$.

b. $c = 8$ is the longest side, so C is the largest angle. But $C = 58.7^\circ$ would leave $B = 180^\circ - 40^\circ - 58.7^\circ = 81.3^\circ$, larger than C, though $b = 3$ is the shortest side. (Or: $c^2 = 64 > a^2 + b^2 \approx 45.2$, so C is obtuse.)

c. $\sin C \approx 0.854$ fits both 58.7° and $180^\circ - 58.7^\circ = 121.3^\circ$; the second is right. Safer: find B, opposite the shortest side, which must be acute: $\sin B = \dfrac{3\sin 40^\circ}{6.02} \approx 0.320$, $B \approx 18.7^\circ$; then $C = 180^\circ - 40^\circ - 18.7^\circ = 121.3^\circ$. An angle opposite a side that isn't the longest is always acute, so the sine law gives it without doubt.

MARKING SCHEME

- ✓ a. $a \approx 6.02$.
- ✓ b. C must be the largest angle (or obtuse), and 58.7° makes B larger.
- ✓ c. $B \approx 18.7^\circ$.
- ✓ c. $C \approx 121.3^\circ$.
- ✓ c. Find first the angle opposite a shorter side: it can't be obtuse.

2 Two trees across a river

ANSWER A · 6

Tests: choosing which triangles to solve, and in what order, to reach a distance nobody can measure.

a. In $\triangle PAB$, $\angle APB = 60^\circ$: $AP = \dfrac{100\sin 50^\circ}{\sin 60^\circ} \approx 88.5$ m. In $\triangle QAB$, $\angle AQB = 40^\circ$: $AQ = \dfrac{100\sin 105^\circ}{\sin 40^\circ} \approx 150.3$ m.

b. In $\triangle APQ$, $\angle PAQ = 70^\circ - 35^\circ = 35^\circ$, between the sides found in a: $PQ^2 = 88.46^2 + 150.27^2 - 2 \times 88.46 \times 150.27\cos 35^\circ \approx 7825 + 22581 - 21777 = 8629$, $PQ \approx 92.9$ m.

c. In $\triangle APQ$ at first only the angle at A is known, one part; a triangle needs three. The triangles on the base AB each have a measured side and two angles, so they come first and supply AP and AQ: then APQ has two sides and the angle between.

MARKING SCHEME

- ✓ a. $AP \approx 88.5$ m.
- ✓ a. $AQ \approx 150.3$ m.
- ✓ b. $\angle PAQ = 35^\circ$.
- ✓ b. The cosine law in $\triangle APQ$.
- ✓ b. $PQ \approx 92.9$ m.
- ✓ c. APQ needs two sides first, which only the base triangles give.

3 A slip in the chain

ANSWER

Tests: how an error passes from triangle to triangle in a chain; reading what the numbers show.

a. $\angle C = 49^\circ$: $BC = \dfrac{2\sin 58^\circ}{\sin 49^\circ} \approx 2.247$ km. Then $CD = \dfrac{2.247\sin 64^\circ}{\sin 50^\circ} \approx 2.637$ km.

b. BC stays 2.214 km; $\angle D = 49^\circ$: $CD = \dfrac{2.214\sin 64^\circ}{\sin 49^\circ} \approx 2.637$ km.

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c. In a, the slip was in the first triangle, but CD, in the second, changed by about 0.039 km (1.5%): BC was wrong, and every triangle built on it inherits the error. The slip in b changed only CD. An error near the base passes into every later triangle of the chain (and adds to their own errors), so those angles are measured most often and most carefully. **HA · 5**

MARKING SCHEME

- ✓ a. $BC \approx 2.247$ km.
- ✓ a. $CD \approx 2.637$ km.
- ✓ b. $CD \approx 2.637$ km, BC unchanged.
- ✓ c. The first slip changes every later side, through BC.
- ✓ c. So the angles near the base matter most.

4 Consecutive sides

ANSWER **HA · 5**

Tests: solving a triangle from a relation between its parts, by two laws at once.

a. The largest angle is $2A$, opposite $n+1$: $\dfrac{n+1}{\sin 2A} = \dfrac{n-1}{\sin A}$, so $\dfrac{n+1}{n-1} = \dfrac{2\sin A\cos A}{\sin A} = 2\cos A$, and $\cos A = \dfrac{n+1}{2(n-1)}$.

b. $\cos A = \dfrac{n^2 + (n+1)^2 - (n-1)^2}{2n(n+1)} = \dfrac{n^2+4n}{2n(n+1)} = \dfrac{n+4}{2(n+1)}$.

c. $\dfrac{n+1}{n-1} = \dfrac{n+4}{n+1}$: $(n+1)^2 = (n+4)(n-1)$, $n^2+2n+1 = n^2+3n-4$, so $n=5$: sides **4, 5, 6**.

Check: $\cos A = \dfrac{6}{8} = \dfrac{3}{4}$ both ways, and the largest angle has

$\cos = \dfrac{16+25-36}{40} = \dfrac{1}{8} = 2\times\dfrac{9}{16} - 1 = \cos 2A$. ✓

MARKING SCHEME

- ✓ a. The sine law with the angles A and $2A$.
- ✓ a. $\cos A = \dfrac{n+1}{2(n-1)}$.
- ✓ b. $\cos A = \dfrac{n+4}{2(n+1)}$.
- ✓ c. $n = 5$.
- ✓ c. Sides 4, 5, 6, checked.

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5 The biggest pen

ANSWER A · 6

Tests: area $\frac{1}{2}bc\sin\theta$ as a function of the angle; a limit from the cosine law changing the best choice, whichever sides meet at the post.

a. $\Delta = \frac{1}{2}\times 30\times 40\sin\theta = 600\sin\theta$: largest, **600 m²**, at $\theta = 90°$. The third side is then $\sqrt{900+1600} = 50$ m.

b. The third side $\sqrt{2500-2400\cos\theta} \le 45$: $2400\cos\theta \ge 475$, $\cos\theta \ge 0.198$, $\theta \le 78.6°$. The area grows with θ up to 90°, so the best is $\theta = 78.6°$: $600\sin 78.6° \approx 588$ m².

c. A right angle between 40 m and 45 m needs a third side of $\sqrt{1600+2025} \approx 60.2$ m, and only the 30 m length is left. The same happens for every pair: a right angle would need $\sqrt{30^2+40^2} = 50 > 45$, $\sqrt{30^2+45^2} \approx 54.1 > 40$, $\sqrt{40^2+45^2} \approx 60.2 > 30$. So with any pair at the post the area, $\frac{1}{2}pq\sin\theta$, is largest at the widest angle the third length allows, when it is used in full: the triangle with sides 30, 40 and 45 every time, the same triangle (three sides fix it), **588 m²**. The tempting answer takes the right angle without checking that the third side can close it.

MARKING SCHEME

- ✓ a. $600\sin\theta$; 600 m² at 90°.
- ✓ a. The third side 50 m.
- ✓ b. $\cos\theta \ge 0.198$, $\theta \le 78.6°$.
- ✓ b. About 588 m².
- ✓ c. The right angle needs 60.2 m, more than the 30 m left.
- ✓ c. Every pair is limited the same way, giving the same 30, 40, 45 triangle: 588 m² at most.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	17	10	**27**

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