# Unit 6 · Properties of a Triangle

HARDER PRACTICE QUESTIONS

Full marks 26

Write your answers on separate paper. Show your working. Use a calculator where needed.

## 1 Which triangles fit? [5]

In $\triangle ABC$, $(a^2 + b^2)\sin(A - B) = (a^2 - b^2)\sin(A + B)$.

a. Using the sine law and $\sin^2 A - \sin^2 B = \sin(A + B)\sin(A - B)$, show that $\sin(A - B)(\sin^2 A + \sin^2 B - \sin^2 C) = 0$.

b. Hence show that the triangle is isosceles or right-angled.

c. Tika says: "So the triangle must be isosceles." Give a triangle with whole-number sides that satisfies the condition and shows she is wrong, and check it in the condition.

## 2 The height of a hill [6]

From a point $P$ on level ground, the angle of elevation of the top $T$ of a hill is $30°$. Walking 1 km straight towards the hill to $Q$, the angle of elevation is $45°$.

a. Find the angles of $\triangle PQT$, and use the sine law to show that the height of the hill is $\dfrac{1000 \sin 30° \sin 45°}{\sin 15°}$ m.

b. Show that this is exactly $500(\sqrt{3} + 1)$ m, and find it to the nearest meter.

c. Suppose the $30°$ was really $31°$. Find the height again. Why does an error of only $1°$ change the answer so much? What would you do in the field to make the answer more reliable?

## 3 A field from its sides [5]

A farmer's field $ABCD$ has sides $AB = 25$ m, $BC = 39$ m, $CD = 52$ m and $DA = 60$ m. (The field is made up.)

a. A surveyor measures the diagonal $BD = 65$ m. Find the area of the field in square meters and in ropani (1 ropani = 508.74 m$^2$). What do the numbers tell you about the field's angles at $A$ and $C$?

b. The farmer says the four sides are enough to know the area. His neighbor's field has the same four sides, but there the surveyor measures the angle $DAB = 60°$ instead of a diagonal. Find the diagonal $BD$ and the area of that field.

c. Was the farmer right? Why does a triangle need nothing more than its three sides, while a four-sided field does?

## 4 Half angles in arithmetic progression [6]

a. Using the half-angle laws and Heron's formula, show that $\cot\dfrac{A}{2} = \dfrac{s(s - a)}{\Delta}$.

b. Hence prove that $\cot\dfrac{A}{2}$, $\cot\dfrac{B}{2}$, $\cot\dfrac{C}{2}$ are in arithmetic progression exactly when $a$, $b$, $c$ are.

c. In a triangle, $\cot\dfrac{A}{2} : \cot\dfrac{B}{2} : \cot\dfrac{C}{2} = 3 : 2 : 1$. Find the ratio of its sides, and show that it is right-angled.

## 5 A triangle with a 60° angle [4]

In $\triangle ABC$, $A = 60°$.

a. Show that $a^2 = (b + c)^2 - 3bc$.

b. Using $4bc \le (b + c)^2$, prove that $b + c \le 2a$.

c. When is $b + c = 2a$? Pratima says a triangle with $A = 60°$, $b = 3$ and $c = 7$ has $a = 4.5$. Without the cosine law's value of $a$, show that she is wrong.