Unit 6 · Properties of a Triangle

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 26

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Which triangles fit?

HA · 5

ANSWER

Tests: turning a condition on sides and angles into one on angles alone; both cases of a product equal to zero kept.

a. With $a = k \sin A$, $b = k \sin B$ and dividing by k^2:
$(\sin^2 A + \sin^2 B) \sin (A - B) = (\sin^2 A - \sin^2 B) \sin (A + B) = \sin (A - B) \sin^2 (A + B)$. And $\sin (A + B) = \sin (180° - C) = \sin C$. Moving everything to one side gives $\sin (A - B)(\sin^2 A + \sin^2 B - \sin^2 C) = 0$.

b. Either $\sin (A - B) = 0$: $A - B$ lies between $-180°$ and $180°$, so $A = B$, **isosceles**. Or $\sin^2 A + \sin^2 B = \sin^2 C$: times k^2, $a^2 + b^2 = c^2$, so $C = 90°$, **right-angled**.

c. The 3-4-5 triangle with $a = 3$, $b = 4$, $c = 5$: $C = 90°$, so $A + B = 90°$ and $\sin (A + B) = 1$. $\sin A = \frac{3}{5}$, $\cos A = \frac{4}{5}$, $\sin B = \frac{4}{5}$, $\cos B = \frac{3}{5}$: $\sin (A - B) = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$. Left side $25 \times (-\frac{7}{25}) = -7$; right side $(9 - 16) \times 1 = -7$. ✓ It isn't isosceles. The right angle must be at C, as b found: the same 3-4-5 triangle with the right angle at A or B (for example $a = 4$, $b = 5$, $c = 3$) does not satisfy the condition (left $-\frac{123}{5}$, right $-\frac{27}{5}$).

MARKING SCHEME

✓ a. Sides changed to $k \sin A$, $k \sin B$.
✓ a. $\sin (A + B) = \sin C$ and the factor $\sin (A - B)$ taken out.
✓ b. $A = B$: isosceles.
✓ b. $a^2 + b^2 = c^2$: right-angled.
✓ c. A right triangle that isn't isosceles (3, 4, 5), checked: both sides -7.

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2 The height of a hill

ANSWER HA · 6

Tests: setting up the sine law in an elevation problem; seeing how a small angle in a triangle magnifies errors (Peak XV's 9°).

a. $\angle P = 30°$; $\angle PQT = 180° - 45° = 135°$; $\angle T = 180° - 30° - 135° = 15°$. Sine law: $QT = \dfrac{1000 \sin 30°}{\sin 15°}$. The height is $QT \sin 45° = \dfrac{1000 \sin 30° \sin 45°}{\sin 15°}$.

b. $\sin 15° = \sin(45° - 30°) = \dfrac{\sqrt{6} - \sqrt{2}}{4}$. So

$$h = \frac{1000 \times \frac{1}{2} \times \frac{\sqrt{2}}{2}}{\frac{\sqrt{2}(\sqrt{3}-1)}{4}} = \frac{1000}{\sqrt{3}-1} = 500(\sqrt{3}+1) \approx 1366 \text{ m}.$$

c. With 31°: $\angle T = 14°$, $h = \dfrac{1000 \sin 31° \sin 45°}{\sin 14°} \approx 1505$ m, 139 m (about 10%) more. The angle at the top is small, 15°, and sin 15° is in the denominator: a 1° change in a small angle is a large share of it (1° in 15° is nearly 7%), so the long sides change a lot. In the field: put the two stations farther apart, so the angle at the top is larger, and measure each angle several times and average (as the survey did from several stations).

MARKING SCHEME

- ✓ a. The angles 30°, 135°, 15°.
- ✓ a. QT by the sine law, and $h = QT \sin 45°$.
- ✓ b. $\sin 15° = \dfrac{\sqrt{6} - \sqrt{2}}{4}$ and $500(\sqrt{3}+1)$.
- ✓ b. 1366 m.
- ✓ c. About 1505 m.
- ✓ c. The small angle at the top magnifies errors; a longer base or repeated measurements.

3 A field from its sides

ANSWER A · 5

Tests: Heron's formula and the other laws of the unit combined on a real shape; seeing that four sides don't fix a quadrilateral.

a. $\triangle ABD$: 25, 60, 65, $s = 75$: $\sqrt{75 \times 50 \times 15 \times 10} = 750$. $\triangle BCD$: 39, 52, 65, $s = 78$: $\sqrt{78 \times 39 \times 26 \times 13} = 1014$. Total **1764 m²**, about **3.47 ropani**. And $25^2 + 60^2 = 4225 = 65^2$, $39^2 + 52^2 = 4225 = 65^2$: the angles at A and C are **right angles** (each triangle's area is then half the product of its short sides, 750 and 1014 ✓).

b. Cosine law in $\triangle ABD$: $BD^2 = 25^2 + 60^2 - 2 \times 25 \times 60 \cos 60° = 4225 - 1500 = 2725$, $BD \approx 52.2$ m. $\triangle ABD = \dfrac{1}{2} \times 25 \times 60 \sin 60° = 375\sqrt{3} \approx 649.5$; $\triangle BCD$ (39, 52, 52.2) by Heron, $s \approx 71.6$: ≈ 942.1. Total about **1592 m²** (3.13 ropani), some 172 m² less.

c. **No.** Three sides fix a triangle: the cosine law gives every angle from them, so a frame of three sticks can't change shape. Four sticks can be pushed out of shape at the corners, and the area changes with it (a and b), so a surveyor needs a diagonal or an angle as well. The tempting shortcut, a "Heron's formula" with the four sides, $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ with $s = 88$, gives 1764: that is Brahmagupta's formula, true only for a field whose corners lie on a circle, as the first field's do (its angles at A and C add up to 180°). It doesn't give the neighbor's field.

MARKING SCHEME

- ✓ a. 750 + 1014 = 1764 m², about 3.47 ropani.
- ✓ a. $25^2 + 60^2 = 39^2 + 52^2 = 65^2$: right angles at A and C.
- ✓ b. $BD^2 = 2725$, $BD \approx 52.2$ m.
- ✓ b. $375\sqrt{3} + 942.1 \approx 1592$ m².
- ✓ c. Not enough: four sides don't fix a quadrilateral as three fix a triangle; a diagonal or an angle is needed.

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4 Half angles in arithmetic progression

ANSWER HA · 6

Tests: the half-angle laws turned into a tool; "exactly when" proved both ways; working back from the half angles to the sides.

a. $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s(s-a)} = \frac{\Delta}{s(s-a)}$ (multiply above and below by $\sqrt{s(s-a)}$). So $\cot\frac{A}{2} = \frac{s(s-a)}{\Delta}$.

b. The three cotangents are $\frac{s}{\Delta}$ times $s-a$, $s-b$, $s-c$, so they are in AP exactly when $s-a$, $s-b$, $s-c$ are: $2(s-b) = (s-a)+(s-c)$, that is $2b = a+c$, which is a, b, c in AP. Each step can be reversed, so it holds both ways.

c. By a, $(s-a):(s-b):(s-c) = 3:2:1$: $s-a = 3t$, $s-b = 2t$, $s-c = t$. Adding, $3s-(a+b+c) = s = 6t$, so $a = 3t$, $b = 4t$, $c = 5t$: the sides are **3 : 4 : 5**, and $a^2+b^2 = 25t^2 = c^2$, so $C = 90°$. (The tempting answer, sides 3 : 2 : 1, is no triangle at all: $2+1=3$. The sides are in AP, as b says they must be.)

MARKING SCHEME

- ✓ a. $\tan\frac{A}{2} = \frac{\Delta}{s(s-a)}$, so $\cot\frac{A}{2} = \frac{s(s-a)}{\Delta}$.
- ✓ b. The cotangents are in proportion to $s-a$, $s-b$, $s-c$.
- ✓ b. $2(s-b) = (s-a)+(s-c)$, that is $2b = a+c$, both ways.
- ✓ c. $s-a = 3t$, $s-b = 2t$, $s-c = t$, so $s = 6t$.
- ✓ c. Sides $3t$, $4t$, $5t$.
- ✓ c. $a^2+b^2 = c^2$: right-angled at C.

5 A triangle with a 60° angle

ANSWER HA · 4

Tests: the cosine law used to prove an inequality; the equality case.

a. $a^2 = b^2+c^2-2bc\cos 60° = b^2+c^2-bc = (b+c)^2-3bc$.

b. $3bc \le \frac{3}{4}(b+c)^2$, so $a^2 \ge (b+c)^2-\frac{3}{4}(b+c)^2 = \frac{(b+c)^2}{4}$, and $a \ge \frac{b+c}{2}$: $b+c \le 2a$.

c. Equality needs $4bc = (b+c)^2$, that is $(b-c)^2 = 0$, $b = c$: then $A = 60°$ and $b = c$ make the triangle equilateral. Pratima's triangle has $b+c = 10$, so $a \ge 5$: 4.5 is impossible. (In fact $a^2 = 9+49-21 = 37$, $a \approx 6.08$.)

MARKING SCHEME

- ✓ a. $a^2 = (b+c)^2-3bc$.
- ✓ b. $a^2 \ge \frac{(b+c)^2}{4}$, so $b+c \le 2a$.
- ✓ c. Equality only when $b = c$: equilateral.
- ✓ c. $a \ge 5$, so 4.5 is impossible.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	5	21	**26**

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