Unit 6 · Properties of a Triangle

ANSWER KEY · FOR TEACHERS

Full marks 37

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

K · 4

1. opposite
2. sine
3. cosine
4. tangent
5. projection
6. semi-perimeter
7. half-angle
8. Heron's

Each carries 0.5 mark.

2 True or false

U · 3

i. ✓ **True**

ii. ✗ **False**: It is **obtuse**: $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} < 0$.

iii. ✓ **True**

iv. ✗ **False**: With three sides no angle is known, so the sine law can't start; the cosine law can.

v. ✓ **True**

vi. ✗ **False**: That is $\sin \dfrac{A}{2}; \cos \dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$.

Each carries 0.5 mark.

3 Multiple choice

i. K · 1

b. $\dfrac{c}{\sin C}$

The sine law: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$. The first option is the same ratio turned upside down, for another corner.

ii. U · 1

b. $\sqrt{19}$

$a^2 = 9 + 25 - 2 \times 3 \times 5 \times \dfrac{1}{2} = 19$. 7 would need $\cos A = -\dfrac{1}{2}$ (120°); $\sqrt{34}$ forgets the $-2bc \cos A$ term.

iii. U · 1

c. 120°

It faces 13: $\cos \theta = \dfrac{49 + 64 - 169}{2 \times 7 \times 8} = \dfrac{-56}{112} = -\dfrac{1}{2}$, so 120°. 60° comes from dropping the minus sign.

iv. A · 1

b. $12\sqrt{5}$

$s = 12$: $\sqrt{12 \times 5 \times 4 \times 3} = \sqrt{720} = 12\sqrt{5} \approx 26.8$. $6\sqrt{5}$ is half of it.

v. HA · 1

a. isosceles

With $a = k \sin A$, $b = k \sin B$: $\sin A \cos B - \cos A \sin B = 0$, so $\sin(A - B) = 0$ and $A = B$. (The last option answers $a \cos A = b \cos B$, a different condition.)

Science Playground, Katunje, Bhaktapur
Book a class visit: https://science-playground.com/booking

4 Short answers

a. MARKING SCHEME U · 2

✓ At $A = 90°$ the term is 0 and the law is Pythagoras's.

✓ The term makes a^2 smaller for an acute angle and larger for an obtuse one.

b. MARKING SCHEME U · 2

✓ The side opposite A, that is a, must be subtracted.

✓ $\cos A = \frac{1}{8}$.

c. MARKING SCHEME A · 2

✓ $s = 9$ and Heron's formula.

✓ $6\sqrt{6} \approx 14.7\ \text{cm}^2$.

d. MARKING SCHEME A · 3

✓ $ab \cos C = \frac{a^2 + b^2 - c^2}{2}$.

✓ $ac \cos B = \frac{a^2 + c^2 - b^2}{2}$.

✓ The difference simplified to $b^2 - c^2$.

5 Long answers

a. ANSWER A · 5

By the sine law, $\frac{b - c}{a} = \frac{\sin B - \sin C}{\sin A}$.

Sum to product above, and the double angle below: $\sin B - \sin C = 2 \cos \frac{B + C}{2} \sin \frac{B - C}{2}$ and $\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$.

$\frac{B + C}{2} = 90° - \frac{A}{2}$, so $\cos \frac{B + C}{2} = \sin \frac{A}{2}$.

So $\frac{b - c}{a} = \frac{2 \sin \frac{A}{2} \sin \frac{B - C}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}} = \frac{\sin \frac{B - C}{2}}{\cos \frac{A}{2}}$.

Cross-multiplying, $(b - c) \cos \frac{A}{2} = a \sin \frac{B - C}{2}$. ✓

MARKING SCHEME

✓ $\frac{b - c}{a} = \frac{\sin B - \sin C}{\sin A}$.

✓ $\sin B - \sin C = 2 \cos \frac{B + C}{2} \sin \frac{B - C}{2}$.

✓ $\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$.

✓ $\cos \frac{B + C}{2} = \sin \frac{A}{2}$.

✓ The result.

Science Playground, Katunje, Bhaktapur
Book a class visit: https://science-playground.com/booking

b. **ANSWER** A · 8

i. $\angle ACB = 180° - 70° - 60° = 50°$.

ii. $AC = \dfrac{500 \sin 60°}{\sin 50°} = \dfrac{433.0}{0.7660} \approx 565.3$ m.

iii. $BC = \dfrac{500 \sin 70°}{\sin 50°} = \dfrac{469.8}{0.7660} \approx 613.3$ m.

iv. The width is the height from C: $AC \sin 70° \approx 565.3 \times 0.9397 \approx 531.2$ m (or $BC \sin 60°$, the same).

v. $AC \cos 70° + BC \cos 60° \approx 193.3 + 306.7 = 500$ m $= AB$. ✓

MARKING SCHEME

✓ i. 50°.
✓ ii. The sine law set up.
✓ ii. 565.3 m.
✓ iii. 613.3 m.
✓ iv. The width as $AC \sin 70°$.
✓ iv. 531.2 m.
✓ v. The projection law $AB = AC \cos A + BC \cos B$.
✓ v. $193.3 + 306.7 = 500$.

6 Think

ANSWER HA · 3

With $b = k \sin B$, $c = k \sin C$: $b \cos B + c \cos C = k(\sin B \cos B + \sin C \cos C) = \dfrac{k}{2}(\sin 2B + \sin 2C)$.

Sum to product: $\sin 2B + \sin 2C = 2 \sin (B + C) \cos (B - C) = 2 \sin A \cos (B - C)$, since $\sin (B + C) = \sin (180° - A) = \sin A$.

So the left side is $k \sin A \cos (B - C) = a \cos (B - C)$. ✓

MARKING SCHEME

✓ Sides changed to $k \sin B$, $k \sin C$, and $\dfrac{k}{2}(\sin 2B + \sin 2C)$.
✓ $\sin 2B + \sin 2C = 2 \sin (B + C) \cos (B - C)$.
✓ $\sin (B + C) = \sin A$, giving $a \cos (B - C)$.

7 Activity: A triangle from its sides

(not marked)

WHAT STUDENTS SHOULD FIND

About 41.4°, 55.8° and 82.8° (cosines $\dfrac{3}{4}$, $\dfrac{9}{16}$, $\dfrac{1}{8}$), adding to 180°; a careful drawing measures within about 1° of these.

The largest angle is twice the smallest: $\cos 2A = 2 \times \dfrac{9}{16} - 1 = \dfrac{1}{8}$.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	5	9	19	4	**37**

Science Playground, Katunje, Bhaktapur
Book a class visit: https://science-playground.com/booking