Unit 5 · Matrix Based System of Linear Equations

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 28

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 When the determinant is zero

ANSWER HA · 6

Tests: $D = 0$ means no solution **or** infinitely many, and only the equations themselves decide which.

a. $D = k(k^2 - 1) - 1(k - 1) + 1(1 - k) = k^3 - k - 2k + 2 = k^3 - 3k + 2 = (k - 1)^2(k + 2)$. (Check: $k = 1$ and $k = -2$ make it 0.)

b. The three equations are alike, so try $x = y = z$: $(k + 2)x = 1$. For $k \neq 1, -2$ the solution is unique, so it is $x = y = z = \mathbf{\frac{1}{k + 2}}$. (Or by Cramer's rule.)

c. $k = 1$: all three equations are $x + y + z = 1$, one plane: **infinitely many** solutions, $x = s$, $y = t$, $z = 1 - s - t$. Anjila is wrong here.

$k = -2$: adding the three equations gives $0x + 0y + 0z = 3$: **no solution**. She is right here.

So $D = 0$ alone can't tell; the case $k = 1$ has no unique answer, but many.

MARKING SCHEME

✓ a. $D = k^3 - 3k + 2$.

✓ a. Factorized as $(k - 1)^2(k + 2)$.

✓ b. $x = y = z = \frac{1}{k + 2}$.

✓ c. $k = 1$: one plane, infinitely many.

✓ c. $k = -2$: adding gives $0 = 3$, no solution.

✓ c. Her claim fails at $k = 1$.

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2 A curve through measured points

ANSWER HA · 6

Tests: a real model turned into a linear system (the unknowns are the coefficients, not t); consistency as a fact about data, and what it lets you conclude.

a. Four equations in a, b, c: $a + b + c = 25$, $4a + 2b + c = 40$, $9a + 3b + c = 45$, $16a + 4b + c = 42$. For the first three the determinant of the coefficients is $-2 \neq 0$ (no two readings at the same time), so exactly one curve fits them (b), and it gives $h(4) = 40$, not 42. (Or row-reduce the 4×4 augmented matrix: a row $(0\ 0\ 0 \mid c)$ with $c \neq 0$ appears.) The system is inconsistent.

b. First three: subtracting, $3a + b = 15$, $5a + b = 5$, so $a = -5$, $b = 30$, $c = 0$: $h = \mathbf{30}t - \mathbf{5}t^{\mathbf{2}}$. Last three: $4a + 2b + c = 40$, $9a + 3b + c = 45$, $16a + 4b + c = 42$ give $5a + b = 5$, $7a + b = -3$, so $a = -4$, $b = 25$, $c = 6$: $h = \mathbf{-4}t^{\mathbf{2}} + \mathbf{25}t + \mathbf{6}$.

c. At $t = 0$ the ball is in his hand, so $h(0) = c = 0$: a fifth, sure, point (0, 0). The curve through the first three readings has $c = 0$, so (0, 0) and readings 1, 2, 3 lie on one curve, and reading 4 does not. If instead reading 1, 2 or 3 were the wrong one, the other two of them, (0, 0) and reading 4 would lie on one curve; but the only curve through (0, 0) and any two of readings 1–3 is $h = 30t - 5t^2$ (three points with different t fix it), which misses reading 4. So **reading 4** is the wrong one (the curve says 40 m). The tempting answer "the curve through the last three is as good as the first" ignores the start: it puts the ball 6 m above his hand at $t = 0$. Greatest height: $h = 45 - 5(t - 3)^2$, **45 m at 3 s**.

MARKING SCHEME

- ✓ a. The curve through three readings is unique and gives 40 m at 4 s, not 42 (or a row $0 = c \neq 0$).
- ✓ b. $h = 30t - 5t^2$.
- ✓ b. $h = -4t^2 + 25t + 6$.
- ✓ c. $h(0) = 0$ means $c = 0$: the first curve passes through the start, the second doesn't.
- ✓ c. Reading 4 is the wrong one (any other choice leaves a curve through the start that misses reading 4).
- ✓ c. Greatest height 45 m, at 3 s.

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3 Two industries that buy from each other

ANSWER A · 6

Tests: building a system from a real setting; what the columns and rows of an inverse mean, used to answer "what if" questions without solving again.

a. What farming produces, less what farming itself uses of it ($0.4x_1$) and what manufacturing uses ($0.2x_2$), is left for final demand: $x_1 - 0.4x_1 - 0.2x_2 = 50$; in the same way $x_2 - 0.4x_1 - 0.2x_2 = 100$. So $M = \begin{pmatrix} 0.6 & -0.2 \\ -0.4 & 0.8 \end{pmatrix}$, $|M| = 0.48 - 0.08 = 0.4$,

$$M^{-1} = \frac{1}{0.4}\begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} = \begin{pmatrix} 2 & 0.5 \\ 1 & 1.5 \end{pmatrix}$$

and $X = M^{-1}F = \begin{pmatrix} 100 + 50 \\ 50 + 150 \end{pmatrix}$: farming **Rs 150 crore**, manufacturing **Rs 200 crore**. (Check: $150 - 60 - 40 = 50$. ✓)

b. The extra outputs are M^{-1} times the change, that is 10 times a column of M^{-1}. Farm demand: (20, 10), Rs 30 crore in all. Manufactured demand: (5, 15), Rs 20 crore in all. **The extra farm demand needs more**, Rs 10 crore more than the other. Smriti's Rs 10 crore is too small both ways: each rupee delivered takes inputs, which take inputs in turn. Farming needs more because it is the heavier user of inputs (Rs 0.80 for each rupee, against Rs 0.40).

c. Row 1 of $X = M^{-1}F$: $x_1 = 2f_1 + 0.5f_2 \leq 200$ with $f_1 = 50$ gives $0.5f_2 \leq 100$: at most **Rs 200 crore** of manufactured goods (manufacturing then produces $50 + 300 = 350$). The tempting answers are "no limit" (as if farming made nothing for manufacturing) and Rs 150 crore (as if each crore of manufactured demand took a crore of farm output). In fact it takes Rs 0.5 crore, the entry in row 1, column 2 of M^{-1}, so farming's spare Rs 50 crore carries Rs 100 crore more.

MARKING SCHEME

✓ a. The two equations, $0.6x_1 - 0.2x_2 = 50$ and $-0.4x_1 + 0.8x_2 = 100$.

✓ a. $M^{-1} = \begin{pmatrix} 2 & 0.5 \\ 1 & 1.5 \end{pmatrix}$.

✓ a. $x_1 = 150$, $x_2 = 200$.

✓ b. Extra farm demand needs (20, 10), extra manufactured demand (5, 15), from the columns of M^{-1}.

✓ b. Rs 30 crore against Rs 20 crore: the farm choice needs more, and both are more than Smriti's 10.

✓ c. $2(50) + 0.5f_2 \leq 200$: at most Rs 200 crore.

4 One matrix, two right sides

ANSWER HA · 6

Tests: a singular matrix found while working, not by being told; the same A giving no solution or infinitely many depending on B.

a. $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$: the left half has rows (1, 2, 3), (0, 1, 1), (0, 1, 1). Then $R_3 \to R_3 - R_2$ makes the third row of the left half (0, 0, 0): no operation can make a 1 on its diagonal, so the left half can't become I. A has no inverse ($|A| = 0$): row 3 of A is row 2 minus row 1.

b. The same operations on $[A \mid B]$: the constants become 1, $3 - 2 = 1$, $2 - 1 - 1 = 0$. The last row is $0 = 0$. From $y + z = 1$ and $x + 2y + 3z = 1$: with $z = t$, $y = 1 - t$, $x = 1 - 2(1 - t) - 3t = -1 - t$. So $x = \mathbf{-1 - t}$, $y = \mathbf{1 - t}$, $z = t$, any t.

c. Now the constants become 1, 1, $3 - 1 - 1 = 1$: the last row says $0 = 1$. **No solution.**

d. The last row's constant is $b_3 - b_1 - (b_2 - 2b_1) = b_3 - b_2 + b_1$, so there is a solution exactly when $b_3 = b_2 - b_1$: the third equation must follow from the first two just as the third row of A does.

MARKING SCHEME

✓ a. A row of zeros appears in the left half, so there is no inverse.

✓ b. The last row $0 = 0$.

✓ b. $x = -1 - t$, $y = 1 - t$, $z = t$.

✓ c. The last row $0 = 1$: no solution.

✓ d. The last constant is $b_1 - b_2 + b_3$.

✓ d. A solution exactly when $b_3 = b_2 - b_1$.

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5 Two operations at once

ANSWER HA · 4

Tests: why row operations keep the solutions (each can be undone); the difference between two operations in turn and two at once.

a. Adding the equations: $2x = 4$, so $x = 2$, $y = 1$: one solution.

b. His new rows are $R_1 - R_2$ and $R_2 - R_1$, which are negatives of each other: they say the same thing, $2y = 2$, and the information in $R_1 + R_2$ (which gives x) is lost. Done one after the other, the second operation would use the **new** first row: $R_2 \to R_2 - (R_1 - R_2) = 2R_2 - R_1$, and the pair $R_1 - R_2$, $2R_2 - R_1$ still gives back the old rows. Each elementary operation can be undone, so nothing is lost; his two-at-once step can't be undone (no combination of two rows that are negatives of each other gives back $R_1 + R_2$), so a solution set with one point became a whole line.

MARKING SCHEME

✓ a. $x = 2, y = 1$.

✓ b. His two new rows are negatives of each other: one equation is lost.

✓ b. Done in turn, the second operation uses the new row and the old rows can be got back.

✓ b. Elementary operations can be undone; his combined step can't, so solutions were gained.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	6	22	**28**

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