Unit 5 · Matrix Based System of Linear Equations

ANSWER KEY · FOR TEACHERS

Full marks 37

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

1. linear
2. coefficient
3. augmented
4. consistent
5. inconsistent
6. Cramer's
7. inverse
8. elementary

K · 4

Each carries 0.5 mark.

2 True or false

i. ✓ **True**

ii. ✗ **False**: It has either no solution or infinitely many; row operations tell which.

iii. ✓ **True**

iv. ✗ **False**: $X = A^{-1}B$: both sides are multiplied by A^{-1} on the left.

v. ✗ **False**: Its first **column**, the column of the coefficients of x.

vi. ✓ **True**

U · 3

Each carries 0.5 mark.

3 Multiple choice

i. **c.** constants — K · 1

A holds the coefficients, X the unknowns and B the constants on the right of the equations.

ii. **c.** infinitely many solutions — U · 1

The second equation is 3 times the first: the same line. $D = 6 - 6 = 0$, and $D_x = 24 - 24 = 0$, $D_y = 12 - 12 = 0$.

iii. **a.** 2 — U · 1

$D = -2 - 3 = -5$; D_y, with y's column replaced by the constants, is $2(-1) - 8(1) = -10$. $y = \frac{-10}{-5} = 2$. $\frac{1}{2}$ is $\frac{D}{D_y}$, upside down.

iv. **a.** 2 — A · 1

$D = 6k - 12 = 0$ at $k = 2$. Then $D_x = 5 \times 6 - 3 \times 9 = 3 \neq 0$: the lines $2x + 3y = 5$ and $2x + 3y = 4.5$ are parallel.

v. **c.** may have no solution or infinitely many — HA · 1

With three unknowns, all four determinants zero doesn't decide it: $x + y + z = 1$, $x + y + z = 2$, $x + y + z = 3$ has them all zero and no solution, while three copies of $x + y + z = 1$ have infinitely many. Row operations decide.

4 Short answers

a. MARKING SCHEME — U · 2

✓ Multiplying by 5 keeps the same solutions and can be undone.

✓ Multiplying by 0 throws the equation away and can't be undone.

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b. MARKING SCHEME U · 2

✓ Not right: the equations are the same line.

✓ Infinitely many solutions ($D_x = D_y = 0$ too).

c. MARKING SCHEME A · 2

✓ $D = -19, D_x = -38, D_y = -57$.

✓ $x = 2, y = 3$.

d. MARKING SCHEME A · 3

✓ $|A| = 1$.

✓ $A^{-1} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$.

✓ $x = 3, y = 1$.

5 Long answers

a. ANSWER A · 5

The augmented matrix (the last column holds the constants):

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 2 & 1 & -1 & 1 \\ 1 & -1 & 2 & 5 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1} \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & -3 & -11 \\ 0 & -2 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow{R_2 \to -R_2,\ R_3 \to R_3 - 2R_2} \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & 7 & 21 \end{pmatrix} \xrightarrow{R_3 \to \frac{1}{7}R_3} \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 3R_3,\ R_1 \to R_1 - R_2 - R_3} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

So $x = 1, y = 2, z = 3$. Check in the third equation: $1 - 2 + 6 = 5$. ✓

MARKING SCHEME

✓ The augmented matrix.

✓ The first column cleared below the pivot.

✓ The second column: a row (0 0 7 | 21) (or the same after other operations).

✓ Back to $[I \mid X]$.

✓ $x = 1, y = 2, z = 3$.

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b. ANSWER A · 8

i. With x, y, z the prices of rice, lentils and sugar (Rs/kg):

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 470 \\ 520 \\ 690 \end{pmatrix}$$

ii. $|A| = 2(4-1) - 1(2-3) + 1(1-6) = 6 + 1 - 5 = 2$.

iii. Cofactors: $C_{11} = 3$, $C_{12} = 1$, $C_{13} = -5$; $C_{21} = -1$, $C_{22} = 1$, $C_{23} = 1$; $C_{31} = -1$, $C_{32} = -1$, $C_{33} = 3$.

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ -5 & 1 & 3 \end{pmatrix}$$

iv. $$X = A^{-1}B = \frac{1}{2}\begin{pmatrix} 1410 - 520 - 690 \\ 470 + 520 - 690 \\ -2350 + 520 + 2070 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 200 \\ 300 \\ 240 \end{pmatrix} = \begin{pmatrix} 100 \\ 150 \\ 120 \end{pmatrix}$$

Rice **Rs 100**, lentils **Rs 150**, sugar **Rs 120** a kilogram. Check: $300 + 150 + 240 = 690$. ✓

MARKING SCHEME

- ✓ i. The matrix equation.
- ✓ ii. $|A| = 2$.
- ✓ iii. The nine cofactors.
- ✓ iii. The adjoint (the cofactors' transpose).
- ✓ iii. $A^{-1} = \frac{1}{2}\text{adj } A$.
- ✓ iv. $X = A^{-1}B$ multiplied out.
- ✓ iv. 100, 150, 120.
- ✓ iv. A check in one equation.

6 Think

ANSWER HA · 3

Row operations $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, then $R_3 \to R_3 - 2R_2$ leave the last row $(0\ 0\ k-5 \mid 0)$.
If $k \neq 5$, the last row gives $z = 0$ and the system has exactly one solution $(x = 0, y = 1, z = 0)$.
If $k = 5$, the last row is all zeros: infinitely many solutions $(z = t, y = 1 - 2t, x = t)$. The constant in the last row is 0 for every k, so the row $(0\ 0\ 0 \mid c)$ with $c \neq 0$ never appears: there is always a solution.

MARKING SCHEME

- ✓ The last row $(0\ 0\ k-5 \mid 0)$.
- ✓ $k \neq 5$: exactly one solution.
- ✓ Its constant is always 0, so no value of k gives no solution ($k = 5$: infinitely many).

7 Activity: Lines and determinants

(not marked)

WHAT STUDENTS SHOULD FIND

First: $D = -2$, $D_x = -6$, $D_y = -2$; the lines cross at (3, 1), one solution. Second: $D = 0$, $D_x = 2$, $D_y = -2$; parallel lines, no solution. Third: $D = D_x = D_y = 0$; the same line, infinitely many. The lines meet at exactly one point only when $D \neq 0$.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	5	9	19	4	**37**

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