Unit 4 · Sequence and Series

HARDER PRACTICE QUESTIONS

Full marks 28

Write your answers on separate paper. Show your working.

1 Sums of sums [6]

a. Find the sum of the first n terms of $1 + (1 + 2) + (1 + 2 + 3) + \cdots$. Arjun says the answer is $\frac{n(n+1)}{2}$. What has he done wrong?

b. Find the sum of the first n terms of $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \cdots$, and check it for $n = 3$.

2 Induction beyond sums [7]

a. Prove by mathematical induction that $7^n - 3^n$ is divisible by 4 for every natural number n.

b. Find the natural numbers n from 1 to 7 for which $2^n > n^2$.

c. Prove by induction that $2^n > n^2$ for every $n \geq 5$. Where does your inductive step use the fact that k is large?

3 A sum that telescopes [5]

$S_n = \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{n(n+1)(n+2)}$. No formula of this unit sums it directly.

a. Example 4 summed $\frac{1}{k(k+1)}$ by writing it as a difference of two terms of the same kind. Write $\frac{1}{k(k+1)(k+2)}$ as a number times the difference of two terms of the kind $\frac{1}{m(m+1)}$, and check your identity.

b. Hence find S_n.

c. What number does S_n get closer to as n grows? Can it ever reach it?

d. How many terms must be added before S_n is within 0.001 of that number?

4 Stacking oranges [5]

A fruit seller stacks oranges on a rectangular tray. The bottom layer is 10 oranges by 15. Each layer above has one orange fewer each way, every orange resting in the hollow between four below, up to a single row on top.

a. How many layers are there, and what is the top layer? How many oranges does the stack hold?

b. Show that a stack whose bottom layer is n by $n + 5$ holds $\frac{n(n+1)(n+8)}{3}$ oranges.

c. She has 1000 oranges and trays of every size n by $n + 5$. Her son reasons: "a pyramid of oranges holds about $\frac{n^3}{3}$, and $\frac{14^3}{3} \approx 915$, so she can build 14 layers." Find the largest stack she can really build, and explain why his estimate is so far out.

5 Odd numbers that make cubes [5]

The odd numbers are grouped one, two, three … at a time:
1 | 3 + 5 | 7 + 9 + 11 | 13 + 15 + 17 + 19 | …

a. How many odd numbers are there in the first $n - 1$ groups? Hence show that the nth group starts with $n^2 - n + 1$.

b. Show that the numbers in the nth group add up to n^3.

c. Using only the sum of the first m odd numbers, m^2, deduce that $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.