# Unit 4 · Sequence and Series

**HARDER PRACTICE QUESTIONS · ANSWER KEY**

Full marks 28

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

## 1 Sums of sums

**ANSWER** A · 6

Tests: finding the $k$th term of a series whose terms are sums, before summing.

a. The $k$th term is $1 + 2 + \cdots + k = \dfrac{k(k+1)}{2} = \dfrac{k^2+k}{2}$. So

$$S_n = \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right] = \frac{n(n+1)}{12}(2n+4) = \frac{n(n+1)(n+2)}{6}$$

Arjun summed $1 + 2 + \cdots + n$, the **last term**, not the sum of all $n$ terms. (Check $n = 3$: $1 + 3 + 6 = 10 = \dfrac{3 \cdot 4 \cdot 5}{6}$.)

b. The $k$th term is $\dfrac{k(k+1)(2k+1)}{6} = \dfrac{2k^3+3k^2+k}{6}$. So

$$S_n = \frac{1}{6}\left[2\left(\frac{n(n+1)}{2}\right)^2 + 3 \cdot \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right] = \frac{n(n+1)}{12}\left[n(n+1) + (2n+1) + 1\right] = \frac{n(n+1)^2(n+2)}{12}$$

Check $n = 3$: $1 + 5 + 14 = 20$, and $\dfrac{3 \times 16 \times 5}{12} = 20$. ✓

**MARKING SCHEME**

✓ a. $t_k = \dfrac{k(k+1)}{2}$.

✓ a. $\dfrac{n(n+1)(n+2)}{6}$.

✓ a. Arjun found only the $n$th term.

✓ b. $t_k = \dfrac{2k^3+3k^2+k}{6}$.

✓ b. $\dfrac{n(n+1)^2(n+2)}{12}$.

✓ b. The check: 20 both ways.

## 2 Induction beyond sums

**ANSWER**

Tests: aiming the inductive step (add and subtract); a base that is not 1; seeing exactly where a step needs its condition.

a. Step 1. $n = 1$: $7 - 3 = 4$. True.

Step 2. Assume $7^k - 3^k = 4m$ for a whole number $m$.

Step 3. $7^{k+1} - 3^{k+1} = 7 \cdot 7^k - 7 \cdot 3^k + 7 \cdot 3^k - 3 \cdot 3^k = 7(7^k - 3^k) + 4 \cdot 3^k = 4(7m + 3^k)$: divisible by 4.

Step 4. By induction, true for every $n$.

b. $n = 1$: $2 > 1$ yes; $n = 2, 3, 4$: $4 = 4$, $8 < 9$, $16 = 16$, no; $n = 5, 6, 7$: $32 > 25$, $64 > 36$, $128 > 49$, yes. So $n = 1, 5, 6, 7$.

c. Step 1. $n = 5$: $32 > 25$. True.

Step 2. Assume $2^k > k^2$ for some $k \geq 5$.

Step 3. $2^{k+1} = 2 \cdot 2^k > 2k^2$. Now $2k^2 \geq (k+1)^2$ when $k^2 - 2k - 1 \geq 0$, that is $(k-1)^2 \geq 2$, true for $k \geq 3$. So $2^{k+1} > (k+1)^2$.

Step 4. True for 5, and for $k + 1$ whenever for $k \geq 5$: true for every $n \geq 5$.

The step uses $k \geq 3$ in $2k^2 \geq (k+1)^2$ (it fails at $k = 2$: $8 < 9$). The base must be 5, not 1: the statement is true at 1, but the chain from 1 breaks at once, as b shows.



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**MARKING SCHEME** HA · 7

✓ a. The base, $7 - 3 = 4$.

✓ a. $7^{k+1} - 3^{k+1} = 7(7^k - 3^k) + 4 \cdot 3^k$.

✓ a. Both parts divisible by 4, so the conclusion.

✓ b. $n = 1, 5, 6, 7$.

✓ c. Base $n = 5$ and the hypothesis for $k \geq 5$.

✓ c. $2^{k+1} > 2k^2 \geq (k+1)^2$.

✓ c. $2k^2 \geq (k+1)^2$ needs $k \geq 3$; the chain can't start at 1.

## 3 A sum that telescopes

**ANSWER** HA · 5

Tests: telescoping when no formula for $\Sigma\, k^r$ fits, with the split found, not given; the meaning of the sum as $n$ grows.

a. The neighbors of $k(k+1)(k+2)$ of the kind $m(m+1)$ are $k(k+1)$ and $(k+1)(k+2)$:
$\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)} = \frac{(k+2)-k}{k(k+1)(k+2)} = \frac{2}{k(k+1)(k+2)}$. So
$\frac{1}{k(k+1)(k+2)} = \frac{1}{2}\left[\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}\right]$. The tempting split,
$\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}$ with no $\frac{1}{2}$, is twice too big (at $k = 1$: $\frac{1}{2} - \frac{1}{6} = \frac{1}{3}$, not $\frac{1}{6}$).

b. Adding for $k = 1$ to $n$, every middle term cancels:

$$S_n = \frac{1}{2}\left[\frac{1}{1 \cdot 2} - \frac{1}{(n+1)(n+2)}\right] = \frac{1}{4} - \frac{1}{2(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

Check $n = 1$: $\frac{4}{24} = \frac{1}{6}$. ✓

c. $\frac{1}{2(n+1)(n+2)}$ shrinks towards 0 as $n$ grows, so $S_n$ closes in on $\frac{1}{4}$. It never reaches it: something positive is always taken away.

d. $\frac{1}{4} - S_n = \frac{1}{2(n+1)(n+2)} < 0.001$ needs $(n+1)(n+2) > 500$. $n = 20$: $21 \times 22 = 462$, not enough; $n = 21$: $22 \times 23 = 506$. So **21 terms**.

**MARKING SCHEME**

✓ a. The split $\frac{1}{2}\left[\frac{1}{k(k+1)} - \frac{1}{(k+1)(k+2)}\right]$, with the factor $\frac{1}{2}$ checked.

✓ b. The sum telescopes to $\frac{1}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right]$.

✓ b. $\frac{n(n+3)}{4(n+1)(n+2)}$ (or the form $\frac{1}{4} - \frac{1}{2(n+1)(n+2)}$).

✓ c. It approaches $\frac{1}{4}$ and never reaches it.

✓ d. $(n+1)(n+2) > 500$: 21 terms.

## 4 Stacking oranges

**ANSWER**

Tests: finding the $k$th term of a real stack before summing; the sum formulas used backwards, and an estimate that fails when the lower powers are large.

a. The short side runs out first: layers of $10 \times 15$, $9 \times 14$, ..., so **10 layers**, the top one a row of $1 \times 6$. Counting from the top, the $k$th layer is $k \times (k+5)$, so the stack holds the sum of $k^2 + 5k$ for $k = 1$ to 10: $385 + 5 \times 55 = 660$ oranges. (The tempting 15 layers, or a top of one orange, treat the tray as square.)

b. Summing $k^2 + 5k$ for $k = 1$ to $n$:
$\frac{n(n+1)(2n+1)}{6} + \frac{5n(n+1)}{2} = \frac{n(n+1)}{6}(2n+1+15) = \frac{n(n+1)(n+8)}{3}$. ✓ (Check $n = 10$: $\frac{10 \times 11 \times 18}{3} = 660$.)



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c. $n = 11$: $\frac{11 \times 12 \times 19}{3} = 836$; $n = 12$: $\frac{12 \times 13 \times 20}{3} = 1040$, too many. So **11 layers** (a tray of 11 by 16), 836 oranges, 164 left. His 14 layers would need $\frac{14 \times 15 \times 22}{3} = 1540$. The stack is $\frac{n^3 + 9n^2 + 8n}{3} = \frac{n^3}{3} + 3n^2 + \frac{8n}{3}$: $\frac{n^3}{3}$ is only the leading term, and here $3n^2$ (the 5 extra oranges in every row, and the steps of the layers) is nearly as big as $\frac{n^3}{3}$ (at $n = 11$: 363 against 444). The estimate is good only when $n$ is large compared with those extras. **A · 5**

**MARKING SCHEME**

✓ a. 10 layers, the top a row of $1 \times 6$.

✓ a. The $k$th layer $k(k + 5)$, and 660 oranges.

✓ b. $\frac{n(n+1)(2n+1)}{6} + \frac{5n(n+1)}{2}$ factorized to $\frac{n(n+1)(n+8)}{3}$.

✓ c. 11 layers (836), since 12 need 1040.

✓ c. The $3n^2$ term is comparable with $\frac{n^3}{3}$ at this size, so the estimate is far too high.

## 5 Odd numbers that make cubes

**ANSWER** **HA · 5**

Tests: a second route to $\Sigma\, k^3$; the general term of a pattern found and used.

a. $1 + 2 + \cdots + (n-1) = \frac{(n-1)n}{2}$ odd numbers. The $m$th odd number is $2m - 1$, so the next one, number $\frac{(n-1)n}{2} + 1$, is $(n-1)n + 2 - 1 = n^2 - n + 1$. (Check: the 4th group starts with $16 - 4 + 1 = 13$. ✓)

b. The group has $n$ odd numbers, an AP with $d = 2$, from $n^2 - n + 1$ to $n^2 - n + 1 + 2(n-1) = n^2 + n - 1$. Its sum is $\frac{n}{2}[(n^2 - n + 1) + (n^2 + n - 1)] = \frac{n}{2} \times 2n^2 = n^3$. ✓

c. The first $n$ groups are the first $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ odd numbers, whose sum is $\left[\frac{n(n+1)}{2}\right]^2$. By b the same numbers add up group by group to $1^3 + 2^3 + \cdots + n^3$. So the two are equal. ∎

**MARKING SCHEME**

✓ a. $\frac{(n-1)n}{2}$ odd numbers before the $n$th group.

✓ a. Its first number $n^2 - n + 1$.

✓ b. Last number $n^2 + n - 1$, and the AP's sum $n^3$.

✓ c. The first $n$ groups hold the first $\frac{n(n+1)}{2}$ odd numbers.

✓ c. Their sum $\left[\frac{n(n+1)}{2}\right]^2$ equals $\Sigma\, k^3$.

| Marks by level | Knowledge | Understanding | Application | Higher ability | Total |
|---|---|---|---|---|---|
| **Marks** | 0 | 0 | 11 | 17 | **28** |



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