Unit 4 · Sequence and Series

ANSWER KEY · FOR TEACHERS

Full marks 37

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

K · 4

1. telescope
2. sum
3. induction
4. base
5. hypothesis
6. inductive
7. cubes
8. square

Each carries 0.5 mark.

2 True or false

U · 3

i. ✓ **True**

ii. ✗ **False**: That is true of the **cubes**. For $n = 2$: $1 + 4 = 5$, but $3^2 = 9$.

iii. ✗ **False**: It is $5 + 5 + \cdots + 5$, n times: $5n$.

iv. ✓ **True**

v. ✗ **False**: The base step is needed too: a step that works proves nothing if the statement is false for $n = 1$.

vi. ✓ **True**

Each carries 0.5 mark.

3 Multiple choice

i. K · 1

b. $\dfrac{n(n+1)}{2}$

An AP with first term 1 and last term n: $\frac{n}{2}(1 + n)$. n^2 is the sum of the odd numbers, $n(n + 1)$ of the even ones.

ii. U · 1

a. 440

$2\Sigma\, k + \Sigma\, 1 = 2 \times 210 + 20 = 440$. 421 adds the 1 only once; 441 would also count the first odd number, 1.

iii. U · 1

b. $2n(3n + 1)$

First factors 2, 4, 6, …: $2n$. Second factors 4, 7, 10, …: an AP with $a = 4$, $d = 3$, so $4 + 3(n - 1) = 3n + 1$.
Check $n = 3$: 6×10. ✓

iv. A · 1

b. 12

$\dfrac{12 \times 13 \times 25}{6} = 650$. The sum is about $\frac{n^3}{3}$, so $n^3 \approx 1950$ and n is near 12.5: try 12.

v. HA · 1

a. $\dfrac{n(n+1)(n+2)}{6}$

The kth term is $k(n + 1 - k)$, so the sum is
$(n + 1)\Sigma\, k - \Sigma\, k^2 = \frac{n(n+1)^2}{2} - \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(n+2)}{6}$. Check $n = 3$: $3 + 4 + 3 = 10$. ✓
Here n is fixed and k runs; the third option forgets the $-\Sigma\, k^2$.

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4 Short answers

a. MARKING SCHEME — U · 2

- ✓ The base step is the first domino falling; the inductive step is each falling domino knocking the next.
- ✓ Without either one, not every domino is sure to fall.

b. MARKING SCHEME — U · 2

- ✓ $\Sigma_{k=1}^{10} 3 = 3 \times 10 = 30$.
- ✓ 415.

c. MARKING SCHEME — A · 2

- ✓ The squares to 30 (9455) less those to 20 (2870).
- ✓ 6585.

d. MARKING SCHEME — A · 3

- ✓ $t_k = 3k^2 + k$.
- ✓ The sums of k^2 and k used.
- ✓ $n(n+1)^2$, checked.

5 Long answers

a. ANSWER — A · 5

Let $P(n)$ be the statement.

Step 1. $n = 1$: the left side is 2 and the right side $3 - 1 = 2$. $P(1)$ is true.

Step 2. Assume $P(k)$: $2 + 6 + \cdots + 2 \cdot 3^{k-1} = 3^k - 1$.

Step 3. Add the next term, $2 \cdot 3^k$: the sum is $3^k - 1 + 2 \cdot 3^k = 3 \cdot 3^k - 1 = 3^{k+1} - 1$, which is $P(k+1)$.

Step 4. $P(1)$ is true and $P(k)$ implies $P(k+1)$, so by induction $P(n)$ is true for every natural number n.

MARKING SCHEME

- ✓ The base step, $n = 1$.
- ✓ The hypothesis $P(k)$ written.
- ✓ The $(k+1)$th term $2 \cdot 3^k$ added.
- ✓ $3^k - 1 + 2 \cdot 3^k = 3^{k+1} - 1$.
- ✓ The conclusion by the principle of induction.

b. ANSWER — A · 8

i. $t_n = (2n-1)(2n+1) = 4n^2 - 1$.

ii. $S_n = 4 \cdot \dfrac{n(n+1)(2n+1)}{6} - n = \dfrac{n}{3}[2(n+1)(2n+1) - 3] = \dfrac{n(4n^2+6n-1)}{3}$.

iii. $3 + 15 = 18$, and $\dfrac{2(16+12-1)}{3} = 18$. ✓

iv. $\dfrac{10(400+60-1)}{3} = 1530$.

v. $n = 8$: $\dfrac{8 \times 303}{3} = 808$; $n = 9$: $\dfrac{9 \times 377}{3} = 1131$. So **9 terms**.

MARKING SCHEME

- ✓ i. $t_n = 4n^2 - 1$.
- ✓ ii. $4\Sigma\, k^2 - \Sigma\, 1$.
- ✓ ii. $\dfrac{n(4n^2+6n-1)}{3}$.
- ✓ iii. 18 both ways.
- ✓ iv. 1530.
- ✓ v. 808 for 8 terms.
- ✓ v. 1131 for 9 terms.
- ✓ v. 9 terms.

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6 Think

ANSWER HA · 3

If $1 + \cdots + k = \dfrac{k^2 + k + 2}{2}$, adding $k + 1$ gives $\dfrac{k^2 + k + 2 + 2k + 2}{2} = \dfrac{(k+1)^2 + (k+1) + 2}{2}$: the formula for $k + 1$. So the step works.

But at $n = 1$ the right side is $\dfrac{4}{2} = 2$, not 1: the base step fails, so the chain never starts. (It is 1 more than the true sum for every n.)

MARKING SCHEME

- ✓ The step: $\dfrac{k^2 + k + 2}{2} + (k + 1) = \dfrac{(k+1)^2 + (k+1) + 2}{2}$.
- ✓ At $n = 1$ it gives 2, not 1.
- ✓ So the base step fails and nothing is proved.

7 Activity: Cubes and squares

(not marked)

WHAT STUDENTS SHOULD FIND

1, 3, 6, 10, 15, 21; squares 1, 9, 36, 100, 225, 441; the sums of cubes are the same, 1, 9, 36, 100, 225, 441. The table only checks six cases; the proof for every n needs 4.3's argument or induction.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	5	9	19	4	**37**

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