Unit 3 · Complex Numbers

HARDER PRACTICE QUESTIONS

Full marks 30

Write your answers on separate paper. Show your working. Arguments are principal arguments, in $-\pi < \theta \leq \pi$; ω is a complex cube root of unity.

1 Arguments that wrap round [5]

Let $z = -1 + i$ and $w = -\sqrt{3} - i$.

a. Find the principal arguments of z and w.

b. Find the principal arguments of zw, $\frac{z}{w}$ and z^4, without multiplying out.

c. Laxmi says "the argument of a product is always the sum of the arguments". Using your answers, say when the principal argument of zw is the sum of the principal arguments of z and w, and when it is not.

2 A square from one corner [5]

A square drawn in the Argand plane has its center at the origin, and one corner at $3 + 4i$.

a. Find the other three corners. Explain your method.

b. The four corners are the four roots of one equation $z^4 = c$. Find c in the form $a + bi$.

c. Find the length of a side of the square, and its area.

3 From De Moivre to an exact product [7]

De Moivre's theorem gives $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

a. Show that the roots of $8x^3 - 6x - 1 = 0$ are $\cos 20°$, $\cos 100°$ and $\cos 140°$.

b. Hence, without a calculator, show that $\cos 20° \cos 40° \cos 80° = \frac{1}{8}$.

c. Show that $\cos 20° + \cos 100° + \cos 140° = 0$ in two ways: from the equation, and by seeing $e^{i20°}$, $e^{i140°}$ and $e^{i260°}$ as the three cube roots of one complex number. Which number?

4 The neutral wire [5]

In a three-phase supply, the currents in the three phases can be written as complex numbers (phasors) $I_1 = I$, $I_2 = I\omega^2$ and $I_3 = I\omega$ when the load is balanced. The current in the neutral wire is their sum, $I_1 + I_2 + I_3$.

a. Show that the neutral carries no current when the load is balanced.

b. In a building, the currents are $I_1 = 10$ A, $I_2 = 10\omega^2$ A and $I_3 = 8\omega$ A (one phase lightly loaded). Find the size of the neutral current.

c. Sabina says: "The three currents are 10 A, 10 A and 8 A, so the neutral carries 28 A." Bhim says: "It carries 0 A, as always." Explain what each has got wrong.

5 A power of sine from Euler's formula [4]

Using $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$, show that $\sin^4\theta = \frac{3 - 4\cos 2\theta + \cos 4\theta}{8}$. Check your result at $\theta = \frac{\pi}{4}$.

6 How many solutions? [4]

Find every complex number z with $z^2 = \bar{z}$, the conjugate of z. Kumar says there are two, because the equation is a quadratic. How many are there, and why is his reason wrong?