Unit 3 · Complex Numbers

HARDER PRACTICE QUESTIONS · ANSWER KEY

Full marks 30

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Arguments that wrap round

ANSWER A · 5

Tests: the quadrant of the argument (not $\tan^{-1}$ alone), and that adding arguments may go past $\pm\pi$.

a. z is in quadrant II with $\tan^{-1} 1 = \frac{\pi}{4}$: $\arg z = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$. w is in quadrant III with $\tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$: $\arg w = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$.

b. zw: $\frac{3\pi}{4} - \frac{5\pi}{6} = -\frac{\pi}{12}$, already in range: $\mathbf{-\frac{\pi}{12}}$.

$\frac{z}{w}$: $\frac{3\pi}{4} + \frac{5\pi}{6} = \frac{19\pi}{12}$, more than π, so take 2π away: $\mathbf{-\frac{5\pi}{12}}$.

z^4: $4 \times \frac{3\pi}{4} = 3\pi$, the same direction as π: π (indeed $z^4 = -4$).

c. The sum is always an argument of the product, but it is the **principal** argument only when it falls in $-\pi < \theta \leq \pi$. Here it did for zw, and not for $\frac{z}{w}$ or z^4, which needed 2π taken away. So Laxmi is right about "an argument", not about "the principal argument".

MARKING SCHEME

✓ a. $\arg z = \frac{3\pi}{4}$.

✓ a. $\arg w = -\frac{5\pi}{6}$.

✓ b. zw: $-\frac{\pi}{12}$.

✓ b. $\frac{z}{w}$: $\frac{19\pi}{12} - 2\pi = -\frac{5\pi}{12}$; z^4: π.

✓ c. The sum is the principal argument only when it lies in $(-\pi, \pi]$; otherwise add or take away 2π.

2 A square from one corner

ANSWER A · 5

Tests: multiplying by i as a quarter turn; the nth roots of a number as a regular polygon, used backwards.

a. Turning a corner through 90° about the center gives the next corner, and multiplying by i (modulus 1, argument $\frac{\pi}{2}$) turns a point through 90° about 0. So the corners are $3 + 4i$, $i(3 + 4i) = -4 + 3i$, $i^2(3 + 4i) = -3 - 4i$, $i^3(3 + 4i) = 4 - 3i$.

b. Each corner is $i^k(3 + 4i)$, and $(i^k)^4 = 1$, so every corner has the same fourth power, $c = (3 + 4i)^4$. $(3 + 4i)^2 = 9 - 16 + 24i = -7 + 24i$, and $(-7 + 24i)^2 = 49 - 576 - 336i$: $c = -527 - 336i$.

c. Side: the distance from $3 + 4i$ to $-4 + 3i$, $|7 + i| = \sqrt{50} = 5\sqrt{2}$. Area: **50** square units. (Or: the half-diagonal is $|3 + 4i| = 5$, so the diagonal is 10 and the area $\frac{10^2}{2} = 50$.)

MARKING SCHEME

✓ a. Multiplying by i turns through 90° about the origin.

✓ a. $-4 + 3i$, $-3 - 4i$, $4 - 3i$.

✓ b. All four have the same fourth power, $(3 + 4i)^4$.

✓ b. $-527 - 336i$.

✓ c. Side $5\sqrt{2}$, area 50.

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3 From De Moivre to an exact product

ANSWER HA · 7

Tests: a trigonometric equation turned into a polynomial by De Moivre's triple angle; roots of a cubic, their sum and product; the cube roots of a number as a triangle whose arrows cancel.

a. Put $x = \cos\theta$: $8x^3 - 6x - 1 = 2(4\cos^3\theta - 3\cos\theta) - 1 = 2\cos 3\theta - 1$. It is zero when $\cos 3\theta = \frac{1}{2}$: $3\theta = 60°, 300°, 420°$, so $\theta = 20°, 100°, 140°$. Their cosines are three different numbers (cosine falls on 0° to 180°), and a cubic has at most three roots, so these are all of them. (The tempting answer stops at $3\theta = 60°$, one root; the other values of 3θ give the other two.)

b. For $ax^3 + bx^2 + cx + d = 0$, the product of the roots is $-\frac{d}{a}$: here $\frac{1}{8}$. So $\cos 20° \cos 100° \cos 140° = \frac{1}{8}$. Now $\cos 100° = -\cos 80°$ and $\cos 140° = -\cos 40°$, and the two minus signs cancel: $\cos 20° \cos 40° \cos 80° = \frac{1}{8}$. ✓

c. From the equation: the sum of the roots is $-\frac{b}{a}$, and there is no x^2 term, so the sum is 0. From the roots: $(e^{i20°})^3 = e^{i60°}$, and the cube roots of $e^{i60°}$ are $e^{i(20° + 120°k)}$, $k = 0, 1, 2$: $e^{i20°}$, $e^{i140°}$, $e^{i260°}$. The number is $e^{i60°} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. They are $e^{i20°}$ times $1, \omega, \omega^2$, so their sum is $e^{i20°}(1 + \omega + \omega^2) = 0$: three equal arrows a third of a turn apart. Its real part is $\cos 20° + \cos 140° + \cos 260° = 0$, and $\cos 260° = \cos 100°$ ($260° = 360° - 100°$). ✓

MARKING SCHEME

✓ a. $8x^3 - 6x - 1 = 2\cos 3\theta - 1$, so $\cos 3\theta = \frac{1}{2}$.

✓ a. $\theta = 20°, 100°, 140°$, three different cosines.

✓ b. The product of the roots is $\frac{1}{8}$.

✓ b. $\cos 100° = -\cos 80°$, $\cos 140° = -\cos 40°$, and the signs cancel.

✓ c. The sum of the roots is 0 (no x^2 term).

✓ c. They are the cube roots of $e^{i60°} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $e^{i20°}(1 + \omega + \omega^2) = 0$.

✓ c. The real part, with $\cos 260° = \cos 100°$.

4 The neutral wire

ANSWER A · 5

Tests: $1 + \omega + \omega^2 = 0$ used, not just recited; complex numbers adding as arrows, not as sizes.

a. $I + I\omega^2 + I\omega = I(1 + \omega + \omega^2) = 0$.

b. $10 + 10\omega^2 + 8\omega = 10(1 + \omega^2) + 8\omega = 10(-\omega) + 8\omega = -2\omega$. The size is $|-2\omega| = 2|\omega| =$ **2 A**.

c. Sabina adds the sizes, as if the three currents pointed the same way; they are 120° apart and mostly cancel. Bhim uses the balanced result, which needs the three sizes equal; here one phase is 2 A short, and exactly that much is left over.

MARKING SCHEME

✓ a. $I(1 + \omega + \omega^2) = 0$.

✓ b. $10(1 + \omega^2) = -10\omega$.

✓ b. Neutral current -2ω, size 2 A.

✓ c. Sabina added sizes of currents 120° apart.

✓ c. Bhim's 0 needs equal currents in all three phases.

5 A power of sine from Euler's formula

ANSWER

Tests: Euler's formula with the binomial theorem, for a power no multiple-angle formula of the unit gives; keeping track of $(2i)^4$ and the middle term.

$$\sin^4\theta = \frac{(e^{i\theta} - e^{-i\theta})^4}{(2i)^4} = \frac{e^{4i\theta} - 4e^{2i\theta} + 6 - 4e^{-2i\theta} + e^{-4i\theta}}{16}$$

(the middle term is $6e^{2i\theta}e^{-2i\theta} = 6$, and $(2i)^4 = 16i^4 = 16$). Pair the terms: $e^{4i\theta} + e^{-4i\theta} = 2\cos 4\theta$ and

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$-4(e^{2i\theta} + e^{-2i\theta}) = -8\cos 2\theta$. So HA · 4

$$\sin^4\theta = \frac{2\cos 4\theta - 8\cos 2\theta + 6}{16} = \frac{3 - 4\cos 2\theta + \cos 4\theta}{8}$$

At $\theta = \frac{\pi}{4}$: left side $\left(\frac{1}{\sqrt{2}}\right)^4 = \frac{1}{4}$; right side $\frac{3 - 0 + (-1)}{8} = \frac{1}{4}$. ✓ (The traps: $(2i)^4 = -16$, from taking $i^4 = -1$, which gives every sign wrong; and dropping the constant 6.)

MARKING SCHEME

- ✓ The binomial expansion of $(e^{i\theta} - e^{-i\theta})^4$, with its middle term 6.
- ✓ $(2i)^4 = 16$.
- ✓ The pairs written as $2\cos 4\theta$ and $-8\cos 2\theta$, giving the result.
- ✓ The check at $\frac{\pi}{4}$.

6 How many solutions?

ANSWER HA · 4

Tests: solving with modulus and argument; an equation with $\bar{z}$ in it is not a polynomial in z.
$z = 0$ is one. Otherwise write $z = r(\cos\theta + i\sin\theta)$, $r > 0$. Then $z^2 = r^2(\cos 2\theta + i\sin 2\theta)$ and $\bar{z} = r(\cos(-\theta) + i\sin(-\theta))$.

Moduli: $r^2 = r$, so $r = 1$. Arguments: $2\theta = -\theta + 2k\pi$, so $\theta = \frac{2k\pi}{3}$: $\theta = 0, \frac{2\pi}{3}, -\frac{2\pi}{3}$.

So $z = 0, 1, \omega, \omega^2$: **four** solutions. (Check: $\omega^2 = \bar{\omega}$.)
Kumar's reason fails because $\bar{z}$ is not a polynomial in z: the equation is really two real equations in x and y, and the rule "a quadratic has two roots" doesn't apply to it.

MARKING SCHEME

- ✓ $z = 0$, and otherwise $r^2 = r$, so $r = 1$.
- ✓ $3\theta = 2k\pi$.
- ✓ The four solutions $0, 1, \omega, \omega^2$.
- ✓ $\bar{z}$ is not a polynomial in z, so "two roots" doesn't apply.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	15	15	**30**

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