Unit 3 · Complex Numbers

ANSWER KEY · FOR TEACHERS

Full marks 37

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

K · 4

1. modulus
2. argument
3. principal
4. polar
5. De Moivre's
6. unity
7. conjugate
8. Euler's

Each carries 0.5 mark.

2 True or false

U · 3

i. ✓ **True**

ii. ✗ **False**: It has three, equally spaced round a circle, 120° apart.

iii. ✓ **True**

iv. ✗ **False**: It is $\cos\theta - i\sin\theta$, that is $\cos(-\theta) + i\sin(-\theta)$.

v. ✓ **True**

vi. ✗ **False**: Multiply their moduli and **add** their arguments.

Each carries 0.5 mark.

3 Multiple choice

i. **a.** 5 — K · 1

$\sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$. 25 forgets the square root; 7 adds the parts.

ii. **b.** $-130°$ — U · 1

The arguments add to 230°, which is past 180°: the same direction is $230° - 360° = -130°$, the principal value.

iii. **a.** 16 — U · 1

$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, so $(1+i)^8 = (\sqrt{2})^8(\cos 2\pi + i\sin 2\pi) = 16$.

iv. **a.** 0 — A · 1

$100 = 3 \times 33 + 1$, so $\omega^{100} = \omega$; $200 = 3 \times 66 + 2$, so $\omega^{200} = \omega^2$. The sum is $\omega + \omega^2 + 1 = 0$.

v. **b.** 4 — HA · 1

The 12th roots are at multiples of 30°, the 8th roots at multiples of 45°. The common angles are the multiples of 90° (the smallest angle that is both): 1, i, -1, $-i$, four of them, the 4th roots of unity.

4 Short answers

a. MARKING SCHEME — U · 2

✓ $-1 - i$ is in quadrant III, and $\frac{\pi}{4}$ is the direction of $1 + i$.

✓ $-\frac{3\pi}{4}$.

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b. MARKING SCHEME U · 2

✓ $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$ (or the correct expansion).

✓ Equating parts gives the double-angle formulas.

c. MARKING SCHEME A · 2

✓ The two moduli and arguments.

✓ $2\sqrt{2}$ and $\frac{5\pi}{12}$.

d. MARKING SCHEME A · 3

✓ $1 - \omega + \omega^2 = -2\omega$.

✓ $1 + \omega - \omega^2 = -2\omega^2$.

✓ $4\omega^3 = 4$.

5 Long answers

a. ANSWER A · 5

$-27 = 27(\cos\pi + i\sin\pi)$. The cube roots are $3\left[\cos\frac{\pi + 2k\pi}{3} + i\sin\frac{\pi + 2k\pi}{3}\right]$, $k = 0, 1, 2$:

$k = 0$: $3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$; $k = 1$: $3(\cos\pi + i\sin\pi) = -3$; $k = 2$: $\frac{3}{2} - \frac{3\sqrt{3}}{2}i$.

All three are 3 from the origin and $\frac{2\pi}{3}$ apart, so they are equally spaced on a circle of radius 3: an equilateral triangle. Their sum is $\frac{3}{2} - 3 + \frac{3}{2} + 0i = 0$.

MARKING SCHEME

✓ -27 in polar form, modulus 27, argument π.

✓ The formula with $\frac{\pi + 2k\pi}{3}$, $k = 0, 1, 2$.

✓ $\frac{3}{2} \pm \frac{3\sqrt{3}}{2}i$.

✓ -3.

✓ Equally spaced on a circle of radius 3, and the sum 0.

b. ANSWER A · 8

i. $r = \sqrt{1+3} = 2$; quadrant IV with $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$, so $\theta = -\frac{\pi}{3}$:

$z = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right] = 2e^{-i\pi/3}$.

ii. $z^{10} = 2^{10}\left[\cos\left(-\frac{10\pi}{3}\right) + i\sin\left(-\frac{10\pi}{3}\right)\right]$. $-\frac{10\pi}{3} + 4\pi = \frac{2\pi}{3}$, so

$z^{10} = 1024\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -512 + 512\sqrt{3}\,i$.

iii. $z^3 = 8[\cos(-\pi) + i\sin(-\pi)] = -8$: the imaginary part is $\sin(-\pi) = 0$.

iv. $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$. Adding: $e^{i\theta} + e^{-i\theta} = 2\cos\theta$. ✓

MARKING SCHEME

✓ i. $r = 2$.

✓ i. $\theta = -\frac{\pi}{3}$ (quadrant IV), in both forms.

✓ ii. 1024 and the angle $-\frac{10\pi}{3}$.

✓ ii. The angle brought to $\frac{2\pi}{3}$.

✓ ii. $-512 + 512\sqrt{3}\,i$.

✓ iii. $z^3 = 8e^{-i\pi} = -8$.

✓ iv. $e^{-i\theta} = \cos\theta - i\sin\theta$.

✓ iv. Adding gives $2\cos\theta$.

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6 Think

ANSWER HA · 3

$(a + b\omega)(a + b\omega^2) = a^2 + ab(\omega + \omega^2) + b^2\omega^3 = a^2 - ab + b^2$, using $\omega + \omega^2 = -1$ and $\omega^3 = 1$.
Then $(a + b)(a^2 - ab + b^2) = a^3 + b^3$. ✓

(Why it had to work: $a^3 + b^3 = 0$ when $\left(-\frac{a}{b}\right)^3 = 1$, that is when $a = -b$, $-b\omega$ or $-b\omega^2$.)

MARKING SCHEME

✓ $(a + b\omega)(a + b\omega^2)$ multiplied out.
✓ $\omega + \omega^2 = -1$, $\omega^3 = 1$ used to get $a^2 - ab + b^2$.
✓ Times $a + b$ gives $a^3 + b^3$.

7 Activity: Multiplying by i is a quarter turn

(not marked)

WHAT STUDENTS SHOULD FIND

$iz = -2 + 3i$, $i^2z = -3 - 2i$, $i^3z = 2 - 3i$. Every distance is $\sqrt{13} \approx 3.6$ cm, and the angles are about 34°, 124°, 214° (or −146°) and 304° (or −56°): each multiplication by i (modulus 1, argument 90°) turns the point a quarter turn and keeps its distance. $i^4z = z$, back where it started.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	5	9	19	4	**37**

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