Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 A binomial with an extra factor

ANSWER A · 4

Tests: using the general term to pair the terms of a product; knowing which powers of x occur at all.

In $\left(x-\frac{2}{x}\right)^8$, $t_{r+1} = C(8,r)x^{8-r}\left(-\frac{2}{x}\right)^r = C(8,r)(-2)^r x^{8-2r}$: only the even powers 8, 6, …, −8 occur.

a. The 1 needs x^0 from the bracket: $r = 4$, $C(8,4)(-2)^4 = 70 \times 16 = 1120$. The x needs x^{-1}, and $8 - 2r = -1$ has no whole-number r. So the term independent of x is **1120**.

b. The 1 needs x^1: impossible (only even powers). The x needs x^0: 1120. The coefficient of x is **1120** too.

c. No. The expansion of the bracket has only even powers of x, so each power in the product comes from just one of 1 and x: an even power from the 1, an odd one from the x.

MARKING SCHEME

✓ The general term $C(8,r)(-2)^r x^{8-2r}$: only even powers.
✓ a. 1120 (from the 1 times the term with $r = 4$).
✓ b. 1120 (from x times the same term).
✓ c. Not right: $8 - 2r = -1$ has no solution, so only one factor gives each power.

2 The greatest coefficient

ANSWER HA · 4

Tests: the ratio of consecutive coefficients; the "greatest in the middle" rule holds only for $(1+x)^n$.

a. $C(10,5) = 252$, the middle (6th) term's.

b. The coefficient of x^r is $a_r = C(10,r)\,2^r$. By $\frac{C(n,r)}{C(n,r-1)} = \frac{n-r+1}{r}$:

$$\frac{a_r}{a_{r-1}} = \frac{11-r}{r} \times 2$$

This is more than 1 while $22 - 2r > r$, that is $r < 7.33$. So the coefficients rise up to $r = 7$ and fall after it: the greatest is $a_7 = C(10,7) \times 2^7 = 120 \times 128 = \mathbf{15{,}360}$ (the 8th term), almost twice 8064.
Bishnu used the rule for $C(10,r)$ alone. The factor 2^r keeps growing, so it pushes the greatest coefficient past the middle.

MARKING SCHEME

✓ a. $C(10,5) = 252$.
✓ b. $\frac{a_r}{a_{r-1}} = \frac{2(11-r)}{r}$.
✓ b. It is more than 1 for $r \leq 7$: greatest at $r = 7$, 15,360.
✓ b. The rule "greatest in the middle" is for $C(n,r)$; the 2^r moves it.

3 Two series multiplied

ANSWER

Tests: two binomial series multiplied, the range of the product, and how many terms a given accuracy needs.

a. $(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots$ and $(1-x)^{-1} = 1 + x + x^2 + \cdots$, both for $|x| < 1$. Multiply, keeping powers up to x^2:

$$1 + \left(1+\frac{1}{2}\right)x + \left(1+\frac{1}{2}-\frac{1}{8}\right)x^2 = 1 + \frac{3}{2}x + \frac{11}{8}x^2 + \cdots$$

Valid where both are: $|x| < 1$.

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b. $x = 0.01$: $1 + 0.015 + \frac{11}{8}(0.0001) = 1.0151375$, so **1.01514**. Enough? The next term of the product has coefficient $1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} = \frac{23}{16} \approx 1.44$ (the coefficients of 1, x, x^2, x^3 in $\sqrt{1+x}$, each times 1 from $(1-x)^{-1}$), so it adds about $1.44 \times 0.000001 = 0.0000014$, and the terms after it far less. That makes 1.0151389, which still rounds to 1.01514: the 5th place stands (the true value is 1.0151390). (The tempting check, "the next term is small, so it can't matter", isn't enough on its own: a value that sits near a rounding edge, like 1.03055 for $x = 0.02$ to 4 places, is decided by that small term.)

c. It needs $x = 0.5$: inside the range, but not small. The left-out terms, about $1.44x^3 \approx 0.18$ and more, are not small, so three terms (2.09) are far from the true value, 2.449.

MARKING SCHEME

✓ a. The two series, each valid for $|x| < 1$.

✓ a. $1 + \frac{3}{2}x + \frac{11}{8}x^2$.

✓ b. 1.0151375, so 1.01514.

✓ b. The next term, about $1.44x^3 = 0.0000014$, leaves 1.0151389: the 5th place doesn't change.

✓ c. $x = 0.5$ is not small: the left-out terms are large.

4 Errors that pull opposite ways

ANSWER HA · 5

Tests: $(1+x)^n \approx 1 + nx$ for errors; a power multiplies an error; errors in opposite directions cancel; the size of what the approximation leaves out.

a. The worked-out volume is $\pi\,(1.02r)^2\,(0.97h) = V(1+0.02)^2(1-0.03) \approx V(1+0.04)(1-0.03) \approx V(1+0.04-0.03) = V(1+0.01)$: about **1% too large**.

b. $1.02^2 \times 0.97 = 1.0404 \times 0.97 = 1.009188$: **0.92% too large**. The estimate leaves out the products of the small numbers: $(0.02)^2 = 0.0004$ from the square, and $0.04 \times (-0.03) = -0.0012$, which together make -0.0008.

c. The radius's error is doubled (it is squared), to about +4%, while the height's, −3%, pulls the other way. They nearly cancel, leaving about 1%. Errors add with their signs, not by size.

MARKING SCHEME

✓ a. $(1.02)^2 \approx 1.04$ (the power doubles the error).

✓ a. $1 + 0.04 - 0.03$: about 1% too large.

✓ b. Exactly 1.009188, 0.92%.

✓ b. The left-out products of small numbers (0.0004 and −0.0012).

✓ c. The errors have opposite signs and partly cancel.

5 Why 72?

ANSWER

Tests: two series used together (log and $(1-x)^{-1}$) in a case the unit doesn't work; judging an approximation by where its left-out terms are small.

a. In n years the money is added to $2n$ times, each time multiplied by $1 + \frac{r}{2}$: it doubles when $\left(1 + \frac{r}{2}\right)^{2n} = 2$, so $n = \dfrac{\log 2}{2\log\left(1 + \frac{r}{2}\right)}$. (The tempting start, $(1+r)^n = 2$ again, ignores the half-yearly adding.) Now $\log\left(1 + \frac{r}{2}\right) = \frac{r}{2} - \frac{r^2}{8} + \cdots = \frac{r}{2}\left(1 - \frac{r}{4} + \cdots\right)$, so $n \approx \frac{0.6931}{r}\left(1 - \frac{r}{4}\right)^{-1} \approx \frac{0.6931}{r}\left(1 + \frac{r}{4}\right) = \frac{0.6931}{r} + \frac{0.6931}{4}$. With $r = \frac{R}{100}$: $n \approx \frac{69.31}{R} + 0.17$. (At 8%: 8.84 years, against 9.01 once a year: adding twice a year saves about 0.17 years, two months, at any small rate.)

b. $\frac{72}{R} = \frac{69.31}{R} + 0.35$ gives $0.35R = 2.69$, $R \approx$ **7.7%**. Near 8% the rule of 72 agrees with the formula.

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c. At 18%: rule of 72, 4.00 years; formula, $3.85 + 0.35 = 4.20$. The **formula** is closer (4.19). At 2%: rule of 72, 36.0; formula, $34.66 + 0.35 = 35.0$. The **formula** again. The rule of 72 replaces the term 0.35 by a fixed nudge, 72 in place of 69.3, which is right only near 8%: below it the rule says too long, above it too short. The formula keeps the r term, so it stays good over a wide range; it only drifts when r is no longer small (at 18% it is out by 0.01). HA · 6

MARKING SCHEME

✓ a. $\left(1 + \frac{r}{2}\right)^{2n} = 2$, so $n = \frac{\log 2}{2 \log (1 + r/2)}$.

✓ a. $\log \left(1 + \frac{r}{2}\right) \approx \frac{r}{2}\left(1 - \frac{r}{4}\right)$ and $\left(1 - \frac{r}{4}\right)^{-1} \approx 1 + \frac{r}{4}$, giving $\frac{69.31}{R} + 0.17$.

✓ b. $0.35R = 2.69$, $R \approx 7.7$.

✓ c. At 18%: 4.00 against 4.20; the formula is closer.

✓ c. At 2%: 36.0 against 35.0; the formula is closer.

✓ c. The rule of 72 is tuned for rates near 8% only.

6 The squares of the coefficients

ANSWER HA · 4

Tests: comparing coefficients of a product, with the symmetry $C(n, r) = C(n, n - r)$; an identity read as a count.

a. On the right the coefficient of x^n is $C(2n, n)$. On the left, x^n comes from x^r in the first bracket (coefficient $C(n, r)$) times x^{n-r} in the second ($C(n, n - r)$), for $r = 0, 1, \ldots, n$: $\Sigma\, C(n, r)\, C(n, n - r)$. Since $C(n, n - r) = C(n, r)$, this is $\Sigma\, C(n, r)^2$. ∎

b. $1 + 16 + 36 + 16 + 1 = 70 = C(8, 4)$. ✓

c. Choose r boys and $4 - r$ girls: $C(4, r)\, C(4, 4 - r) = C(4, r)^2$ ways; adding over $r = 0$ to 4 counts every choice of 4 from 8 once, $C(8, 4)$.

MARKING SCHEME

✓ a. The coefficient of x^n on the left is $\Sigma\, C(n, r)\, C(n, n - r)$.

✓ a. With $C(n, n - r) = C(n, r)$ it is $\Sigma\, C(n, r)^2 = C(2n, n)$.

✓ b. 70.

✓ c. r boys and $4 - r$ girls: $C(4, r)^2$, summed over r.

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	0	0	9	19	**28**

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