Unit 2 · Binomial Theorem

ANSWER KEY · FOR TEACHERS

Full marks 37

Give the marks for the points in the marking scheme; an answer in other words that says the same is right.

1 Key words

K · 4

1. binomial
2. general
3. middle
4. independent
5. Pascal's
6. index
7. approximation
8. Euler's

Each carries 0.5 mark.

2 True or false

U · 3

i. ✓ **True**

ii. ✗ **False**: Only for $|x| < 1$: with a negative index the series never ends, and outside that range its terms grow.

iii. ✗ **False**: It is $C(n, 4)x^4$: the $(r + 1)$th term has r, so the 5th has $r = 4$.

iv. ✓ **True**

v. ✗ **False**: It equals e^x for every x: $r!$ grows faster than x^r.

vi. ✓ **True**

Each carries 0.5 mark.

3 Multiple choice

i. **b.** 11 — K · 1

The power of y runs 0, 1, …, 10: eleven terms.

ii. **a.** 20 — U · 1

$t_{r+1} = C(6, r)x^{6-2r}$; $6 - 2r = 0$ at $r = 3$: $C(6, 3) = 20$. 15 is $C(6, 2)$, the 3rd term's coefficient.

iii. **b.** -4 — U · 1

$\frac{(-2)(-3)(-4)}{3!} = -4$: $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$. Three negative factors make it negative.

iv. **a.** 0.995 — A · 1

$(1 + 0.01)^{-1/2} \approx 1 - \frac{1}{2}(0.01) = 0.995$ (truly 0.99504). 0.990 forgets the $\frac{1}{2}$; 1.005 forgets the minus sign.

v. **a.** $(1.01)^{100} > 2$ — HA · 1

$(1 + 0.01)^{100} = 1 + 100(0.01) + C(100, 2)(0.01)^2 + \cdots = 2 +$ (positive terms), so it is more than 2 (in fact 2.70). The approximation $1 + nx = 2$ leaves out terms that are all positive.

4 Short answers

a. U · 2

MARKING SCHEME

✓ For a positive integer n, the factor $(n - n) = 0$ makes every term after the $(n + 1)$th zero.

✓ For $n = \frac{1}{2}$ no factor $n - k$ is ever 0.

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b. MARKING SCHEME U · 2

✓ $4^{1/2} = 2$ taken out, so $\approx 2 + \frac{x}{4}$.

✓ Valid for $|x| < 4$.

c. MARKING SCHEME A · 2

✓ The power of x in the general term, $16 - 3r$, so $r = 3$.

✓ 448.

d. MARKING SCHEME A · 3

✓ $2C(n, 2) = C(n, 1) + C(n, 3)$.

✓ $n^2 - 9n + 14 = 0$.

✓ $n = 7$, since $n = 2$ has no 4th term.

5 Long answers

a. ANSWER A · 5

i. $n = -\frac{1}{2}$, with $-x$ for x: $nx \to \frac{x}{2}$; $\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}x^2 = \frac{3}{8}x^2$; $\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(-x)^3 = \frac{5}{16}x^3$.

$(1 - x)^{-1/2} = 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \cdots$, valid for $|x| < 1$.

ii. $1 + 0.01 + 0.00015 + 0.0000025 = 1.0101525$, so $\frac{1}{\sqrt{0.98}} = 1.0102$.

MARKING SCHEME

✓ i. $\frac{x}{2}$ and $\frac{3x^2}{8}$.

✓ i. $\frac{5x^3}{16}$.

✓ i. Valid for $|x| < 1$.

✓ ii. The terms 1, 0.01, 0.00015, 0.0000025 added.

✓ ii. 1.0102.

b. ANSWER A · 8

i. $t_{r+1} = C(9, r)(2x^2)^{9-r}\left(-\frac{1}{3x}\right)^r = C(9, r)2^{9-r}\left(-\frac{1}{3}\right)^r x^{18-3r}$.

ii. $18 - 3r = 0$, $r = 6$: $C(9, 6) \times 2^3 \times \frac{1}{3^6} = \frac{84 \times 8}{729} = \frac{224}{243}$.

iii. $18 - 3r = 3$, $r = 5$: $C(9, 5) \times 2^4 \times \left(-\frac{1}{3^5}\right) = -\frac{126 \times 16}{243} = -\frac{224}{27}$.

iv. Ten terms, so the 5th and 6th ($r = 4, 5$): $C(9, 4) \times 2^5 \times \frac{1}{3^4}x^6 = \frac{448}{9}x^6$ and $-\frac{224}{27}x^3$.

v. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ for every x. $e^{-0.1} = 1 - 0.1 + 0.005 - 0.000167 + 0.000004 = 0.904837$, so 0.9048.

MARKING SCHEME

✓ i. The general term with x^{18-3r}.

✓ ii. $r = 6$.

✓ ii. $\frac{224}{243}$.

✓ iii. $r = 5$, coefficient $-\frac{224}{27}$.

✓ iv. The 5th and 6th terms are the middle ones.

✓ iv. $\frac{448}{9}x^6$ and $-\frac{224}{27}x^3$.

✓ v. The series for e^x.

✓ v. 0.9048.

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6 Think

ANSWER HA · 3

$(1 + 0.1)^{10} = 1 + 10(0.1) + C(10, 2)(0.1)^2 + C(10, 3)(0.1)^3 + \cdots = 1 + 1 + 0.45 + 0.12 + \cdots$. The first four terms already give 2.57, and every later term is positive, so $(1.1)^{10} > 2.57 > 2.5$. (It is 2.594.)

MARKING SCHEME

- ✓ The expansion with $x = 0.1$, $n = 10$.
- ✓ The first terms add up to 2.57 (at least the first three to 2.45 and the fourth 0.12).
- ✓ All the other terms are positive, so it is more than 2.5.

7 Activity: How far can 1 + nx be trusted?

(not marked)

WHAT STUDENTS SHOULD FIND

1.0510, 1.05, 1.051; 1.1046, 1.10, 1.1045; 1.6446, 1.5, 1.6225; 2.7048, 2.0, 2.495. Two terms are good to 2 places up to about $n = 10$; at $n = 50$ they are off by 0.14. The left-out term $\frac{n(n-1)}{2}x^2$ grows like n^2, so "x small" must really mean "nx small".

Marks by level	Knowledge	Understanding	Application	Higher ability	Total
Marks	5	9	19	4	**37**

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